Notes on quantum field theory

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Abstract

Quantum field theory (QFT) is, like the Middle Ages, "enormous and delicate". In trying to beat a path to the heart of it, some tradeoffs are unavoidable. One of then is that we choose to concentrate in the first part of this course more in ideas than in calculations. The theory of spinless fields is employed as a general workshop for developing the main concepts, before tackling the complications of more realistic models.

Also in QFT, tradition = *Schlamperei.* There are lot a pedagogically less-good and even mistaken approaches that have taken root. Now and then we shall debunk them.

Starred sections contain illustrative rather than required material. However, all the exercises are indispensable; and starred exercises mean greater difficulty.

Let us begin by fixing conventions: nearly always $c = 1$ and $\hbar = 1$. This reduces the mass, length and time units to a single one, that we call mass: $M = T^{-1} = L^{-1}$. One can come back to the usual unit system from a result of dimension M^a whose physical units are $L^b T^d M^e$ by multiplying it by $\hbar^{e-a} c^{a-d-e}$.

Contents

1 Relativistic invariance

1.1 Preliminaries

In vacuum light propagates with respect to any inertial system and in all directions with a universal velocity c, which is a constant of nature. This led to the introduction of the Minkowski space M_4, defined as $(\mathbb{R}^4, g)$ with g the Lorentz bilinear form: if $x_1 := (x_1^0 = ct_1, \boldsymbol{x}_1)$, $x_2 = (x_2^0 = ct_2, \boldsymbol{x}_2)$ are vectors in M_4, their Minkowski product is denoted by

$$(x_1 x_2) := x_1^0 x_2^0 - \boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = g(x_1, x_2).$$

Here t is a time coordinate. Recall that units are taken so that $c = 1$. Then one uses the metric tensor

$$g = (g_{\mu\nu}) = (g^{\mu\nu}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

to lower or raise indices: $x_\mu := g_{\mu\nu} x^\nu = (x^0, -\boldsymbol{x})$. A four-vector x is timelike if $(xx) > 0$; spacelike if $(xx) < 0$; lightlike or null if $(xx) = 0$.

The *Poincaré group* $\mathcal{P}$ is by definition the group of transformations of M_4 leaving g invariant; it is then the semidirect product $T_4 \rtimes O(1,3)$, where T_4 denotes the subgroup of spacetime translations and $O(1,3) =: L$ is called the (full) *Lorentz group*, of which of we can think of as of the group of matrices Λ for which $\Lambda^t g \Lambda = g$. We write

$$(a, \Lambda) \cdot (a', \Lambda') = (a + \Lambda a', \Lambda\Lambda') \quad \text{for} \quad a, a' \in T_4,\ \Lambda, \Lambda' \in O(1,3).$$

Also

$$(a, \Lambda)^{-1} = (-\Lambda^{-1} a, \Lambda^{-1}).$$

The laws of nature are invariant under the group of transformations $x \mapsto x' = \Lambda x + a$, in conditions under which the effects of gravity are negligible.

We focus for a while on the Lorentz subgroup. We have identified a general element Λ of it with a 4×4 real matrix $(\Lambda^\mu_{\ \nu})$ so that

$$(\Lambda x)^\mu = \Lambda^\mu_{\ \nu} x^\nu. \tag{1.1}$$

Using the metric tensor g to lower or raise indices, invariance of the form $(xx) = g_{\mu\nu} x^\mu x^\nu$ means $g_{\mu\nu} \Lambda^\mu_{\ \kappa} \Lambda^\nu_{\ \lambda} = g_{\kappa\lambda}$, thus

$$\delta^\nu_\mu \Lambda^\mu_{\ \kappa} \Lambda_\nu^{\ \kappa} = \delta^\kappa_\kappa (= 4), \quad \text{thus} \quad \Lambda^\mu_{\ \kappa} \Lambda_\nu^{\ \kappa} = \delta^\mu_\nu, \quad \text{thus} \quad (\Lambda^{-1})^\kappa_{\ \nu} = \Lambda_\nu^{\ \kappa}.$$

One sees as well that $(\Lambda^{-1})^{\kappa\nu} = \Lambda^{\nu\kappa}$; $\Lambda^\mu_{\ \rho} g^{\rho\sigma} \Lambda^\nu_{\ \sigma} = g^{\mu\nu}$. Since the inverse $\Lambda^{-1} = g\Lambda^t g$ also belongs to the group, we have $\Lambda g \Lambda^t = g$, which says the transpose belongs to the group as well.

Sometimes we use the notation Λ^{-t} for the contragredient matrix. Anticipating future needs, we introduce the notation for partial derivatives with respect to the contravariant and covariant variables:

$$\partial_\mu := \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right); \quad \partial^\mu := \frac{\partial}{\partial x_\mu} = g^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}, -\nabla\right).$$

These notations are natural in that say $\partial_\mu(xp) = \frac{\partial}{\partial x^\mu}(x^\nu p_\nu) = p_\mu$; $\partial^\mu(xp) = \frac{\partial}{\partial x_\mu}(x_\nu p^\nu) = p^\mu$.

Clearly $(\det \Lambda)^2 = 1$. This group has four connected components. To begin with, the determinant of an element of $O(1,3)$ can be $+1$ or -1. Examples of Lorentz transformations with negative determinant are

$$I_s := \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \qquad I_t := \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

the space-reflection and time-reversal transformations. Now, although

$$I_s I_t = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

has determinant $+1$, it cannot be continuously joined to the identity. In effect, if $\Lambda = (\Lambda^\mu{}_\nu)$, from $g(\Lambda x, \Lambda x) = g(x,x)$ for $x = (1, \mathbf{0}) \in M_4$ we infer

$$(\Lambda^0{}_0)^2 = 1 + (\Lambda^1{}_0)^2 + (\Lambda^2{}_0)^2 + (\Lambda^3{}_0)^2,$$

implying that the sign of $\Lambda^0{}_0$ must be constant on any component. There are then four "pieces" in $O(1,3)$, denoted $L^\uparrow_+, L^\downarrow_+, L^\uparrow_-, L^\downarrow_-$, where the index $+$ or $-$ refers to the property $\det \Lambda = +1$ or -1 respectively, and the upwards arrow means $\Lambda^0_0 \geq 1$ whereas the downwards arrow means $\Lambda^0_0 \leq -1$. A typical element of $L^\uparrow_-$ is parity or space reflection, above defined by

$$(I_s x)^0 = x^0; \qquad (I_s \boldsymbol{x}) = -\boldsymbol{x}.$$

We have $I_s{}^0{}_0 = 1$, but $\det I_s = -1$. We see that it maps $L^\uparrow_+$ bijectively into $L^\uparrow_-$. Together they form the *ortochronous* Lorentz group. This is a semidirect product: $L^\uparrow = SO_0(1,3) \ltimes \mathbb{Z}_2$, for $I_s\Lambda \neq \Lambda I_s$ in general. Precisely, the product in $L^\uparrow$ is given by

$$(I_s, \Lambda) \cdot (z, \Lambda') = (I_s z, \Lambda I_s \Lambda' I_s),$$

for z equal to 1 or I_s. Analogously for the *ortochorous* Lorentz group $L^\uparrow_+ \cup L^\downarrow_-$, with the time inversion operator I_t

$$(I_t x)^0 = -x^0; \qquad (I_t \boldsymbol{x}) = \boldsymbol{x}$$

being the typical element of $L^\downarrow_-$; while the *proper* Lorentz group $SO(1,3) \equiv L^\uparrow_+ \cup L^\downarrow_+$ is a direct product. Thus the full or extended Lorentz group has the structure,

$$L = O(1,3) = SO(1,3) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2).$$

The noninvariance of particle physics under space-reflection and time-reversal teaches us that the more relevant group is $SO_0(1,3)$ —or more precisely, its double cover $\mathrm{Spin}_0(1,3)$. Let $\sigma := (1_2, -\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3)$ is the set of Pauli matrices in $\mathbb{C}^{2\times 2}$, let $\bar\sigma := (1_2, \boldsymbol{\sigma})$ for good measure, and let

$$X := (x\sigma) = x^0 1_2 + \boldsymbol{x}\cdot\boldsymbol{\sigma} \quad \text{for} \quad x \in T_4$$

be the corresponding *hermitian* matrix in $\mathbb{C}^{2\times 2}$. It is easily seen that this is a general hermitian matrix, with $x = \frac{1}{2}\operatorname{tr}\big(\bar\sigma X\big)$, and $\det X = (xx)$. If $A \in SL(2,\mathbb{C})$, we call Λ_A its natural image in $SO_0(1,3)$. That is, $\Lambda_A x$ is the 4-vector corresponding to $AXA^\dagger$:

$$A(x\sigma)A^\dagger = \big(\Lambda_A x\,\sigma\big); \tag{1.2}$$

still $\det\big[A(x\sigma)A^\dagger\big] = (xx)$, and we see from an exercise below that $\Lambda_A{}^0{}_0 \geq 0$, so Λ_A belongs in the restricted Lorentz group $SO_0(1,3)$. The map $A \to \Lambda_A$ is extremely important. It is a double covering; at least it is clear from (1.2) that to A and to $-A$ do correspond the same Lorentz transformation. Locally the map is one-to-one. It is clearly a group map, so in particular $\Lambda_{\pm A^{-1}} = \Lambda_{\pm A}^{-1}$. In conclusion $\mathrm{Spin}_0(1,3) \simeq SL(2,\mathbb{C})$.

In particular, since $A\,\sigma_\nu x^\nu A^\dagger = \sigma_\mu \Lambda^\mu{}_\nu x^\nu$ for all x, it must be $A\sigma_\nu A^\dagger = \Lambda^\mu{}_\nu \sigma_\mu$ (omitting A from the notation Λ_A). Analogously,

$$A^{\dagger^{-1}}\bar\sigma_\nu A^{-1} = \Lambda^\mu{}_\nu \bar\sigma_\mu; \qquad A\sigma^\rho A^\dagger = (\Lambda^{-1})^\rho{}_\lambda \sigma^\lambda; \qquad A^{\dagger^{-1}}\bar\sigma^\rho A^{-1} = (\Lambda^{-1})^\rho{}_\lambda \bar\sigma^\lambda.$$

Exercise 1. Work out $I_s \Lambda I_s$ and $I_t \Lambda I_t$ explicitly.

Exercise 2. Prove that the product of two ortochronous transformations is orthochronous.

Exercise 3. Consider the following (important) element of $SL(2,\mathbb{C})$:

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -J^\dagger.$$

Prove that $\Lambda_J = I_1 I_3$, for I_i the reflection with respect to the 3-plane orthogonal to the x_i axis.

Exercise 4. Verify that $\Lambda_A{}^\mu{}_\nu = \frac{1}{2}\operatorname{tr}\,\bar\sigma^\mu A \sigma_\nu A^\dagger$.

Exercise 5. * Prove that $\mathrm{Spin}(1,3) \simeq \mathrm{Spin}(3,1)$. Prove nevertheless that the respective double covers $\mathrm{Pin}(1,3)$ and $\mathrm{Pin}(3,1)$ of $O(1,3)$ and $O(3,1)$ are not isomorphic.

1.2 Rotations and boosts

Typical elements of $SO_0(1,3)$ are the rotations

$$(t', \boldsymbol{x}') = \big(t, \cos\alpha\,\boldsymbol{x} + (1-\cos\alpha)(\boldsymbol{n}\cdot\boldsymbol{x})\,\boldsymbol{n} + \sin\alpha\,\boldsymbol{n}\wedge\boldsymbol{x}\big) =: R_{\alpha\boldsymbol{n}}\,x; \tag{1.3}$$

where $\boldsymbol{n}$ is a unit vector and α is the rotation angle. It is necessary to restrict to $0 \leq \alpha < \pi$ to obtain a one-to-one assignment between rotations and rotation vectors $\alpha\boldsymbol{n}$; to obtain all rotations we must add $\alpha = \pi$, but then the same rotation corresponds to $\boldsymbol{n}$ and $-\boldsymbol{n}$. Note $\operatorname{tr} R_{\alpha\boldsymbol{n}} = 1 + 2\cos\alpha$. If $R = (R_{jk}) = (R^{jk})$ is a (proper) orthogonal matrix, this formula allows to calculate the rotation angle; note $R^{jk} = -\Lambda^{jk}$. Moreover, when $\alpha \neq 0, \pi$, the direction cosines for the axis are obtained from

$$n^i = \frac{-\epsilon^{ijk} R_{jk}}{2\sin\alpha}.$$

The associated interpretation to (1.3) is *active*, if we think of $\boldsymbol{x}', \boldsymbol{x}$ as the components, with respect to an orthonormal basis which remains fixed, respectively of the rotated and original vector. In the passive interpretation, the same formula describes a basis rotated around the same axis in the opposite sense, with vectors fixed. We thus have

$$R_{x,\alpha} = \begin{pmatrix} 1 & & \\ & \cos\alpha & -\sin\alpha \\ & \sin\alpha & \cos\alpha \end{pmatrix} \quad \text{and} \quad R_{z,\alpha} = \begin{pmatrix} \cos\alpha & -\sin\alpha & \\ \sin\alpha & \cos\alpha & \\ & & 1 \end{pmatrix}$$

for counterclockwise rotations of vectors through an angle α respectively around the coordinate axis x, z. As well,

$$R_{y,\alpha} = \begin{pmatrix} \cos\alpha & & \sin\alpha \\ & 1 & \\ -\sin\alpha & & \cos\alpha \end{pmatrix}.$$

Also typical elements of $SO_0(1,3)$ are the (Lorentz) *boosts* or special Lorentz transformations. They describe the situation where two similarly oriented inertial systems whose origins coincide at $t = t' = 0$ move with constant velocity $\boldsymbol{v}$ relative at each other. We consider first the case in which $\boldsymbol{v} = v\boldsymbol{e}_x$. Then $y' = y, z' = z$ and $(t+x)(t-x)$ must be invariant. We must have

$$t' + x' = f(v)(t+x); \quad t' - x' = \frac{1}{f(v)}(t-x),$$

with $f(v) > 0$. Consideration of the origin O' of the primed system gives

$$\Big(f(v) - \frac{1}{f(v)}\Big)t(O') + \Big(f(v) + \frac{1}{f(v)}\Big)x(O') = 0; \quad x(O') = vt(O'),$$

from which

$$f^2(v) - 1 + v(f^2(v)+1) = 0, \quad \text{that is,} \quad f(v) = \sqrt{\frac{1-v}{1+v}}. \tag{1.4}$$

Introducing $\gamma = \frac{1}{\sqrt{1-v^2}}$, we conclude

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f + \frac{1}{f(v)} & f - \frac{1}{f(v)} \\ f - \frac{1}{f(v)} & f + \frac{1}{f(v)} \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix}, \tag{1.5}$$

so

$$\Lambda(-v\boldsymbol{e}_x) =: L(-v\boldsymbol{e}_x) = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This transformation belongs to $L_+^\uparrow$. The reason for the minus sign in the notation $L(-v\boldsymbol{e}_x)$ is that we adopt for boosts the "active" viewpoint, too. For consider the four-vector $(\gamma, \gamma v\boldsymbol{e}_x)$. We have $\Lambda(-v\boldsymbol{e}_x)(\gamma, \gamma v\boldsymbol{e}_x) = (1, \boldsymbol{0})$. Therefore $\Lambda(-v\boldsymbol{e}_x)$ transforms something moving at speed $v\boldsymbol{e}_x$ into something at rest.

Assume now that $\boldsymbol{v}$ does not point along the direction of the x-axis. We can rotate the coordinate system so that the new x-axis points along $\boldsymbol{v}$; perform (1.5); and undo the rotation. The resulting $L(\boldsymbol{v})$ will still be symmetric, so we can write

$$L(-\boldsymbol{v}) = \begin{pmatrix} \gamma & -\gamma v^k \\ -\gamma v^k & T^{ik} \end{pmatrix},$$

with (T^{ik}) symmetric. Now we make the Ansatz $T^{ik} = \delta^{ik} + av^i v^k$. To compute a, we use

$$0 = x'(O') = -\gamma v^i t + T^{ik} x^k(O') = -\gamma v^i t + T^{ik} v^k t.$$

Therefore $1 + av^2 = \gamma$, and $a = \frac{\gamma^2}{\gamma+1}$. Hence

$$L(\boldsymbol{v}) = L(|\boldsymbol{v}|\boldsymbol{n}) = \begin{pmatrix} \gamma & \gamma v^k \\ \gamma v^k & \delta^{ik} + \frac{\gamma^2 v^i v^k}{\gamma+1} \end{pmatrix}. \tag{1.6}$$

Write now $f(v) = \exp(-\zeta v)$. Then $v = \tanh\zeta$ and

$$L(v\boldsymbol{e}_x) = \begin{pmatrix} \cosh\zeta & \sinh\zeta & 0 & 0 \\ \sinh\zeta & \cosh\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then the formula

$$(t', \boldsymbol{x}') = L_{\zeta\boldsymbol{n}}(t, \boldsymbol{x}) := \big(t \cosh\zeta + (\boldsymbol{n}\cdot\boldsymbol{x})\sinh\zeta, \boldsymbol{x} + (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{x})\boldsymbol{n} + t \sin\zeta\boldsymbol{n}\big), \tag{1.7}$$

somewhat in parallel with (1.3), is obtained. In there $\boldsymbol{n}$ is a unit vector again; $0 \le \zeta < \infty$ and the velocity $\boldsymbol{v}$ (with $v = |\boldsymbol{v}| < 1$) of the boost is given by $\boldsymbol{v} = \boldsymbol{n}\tanh\zeta$, so $\cosh\zeta = \gamma$ and $\cosh\zeta - 1 = \gamma - 1 = \gamma^2 v^2 (\gamma+1)^{-1}$.

We close this susbection by noting that the distinction between active and passive transformations is particularly pertinent concerning reversals. Space reversals can be passively performed without problems; it is less obvious, although feasible, how to set up space reversals actively. On the other hand, time reversal can only be realized actively.

Exercise 6. Prove the relation

$$\operatorname{tr} R = \tfrac{1}{2}[(\operatorname{tr} R)^2 - \operatorname{tr} R^2]$$

for rotations R.

Exercise 7. Why is the positive root chosen in (1.4)?

Exercise 8. Prove the linear Doppler effect formula

$$\nu' = f(v)\nu,$$

for the frequencies ν, ν' of a plane light wave as seen by two inertial reference systems whose relative speed is v. The wave is traveling parallely to that speed.

Exercise 9. Consider three inertial reference systems S_1, S_2, S_3 moving in parallel. The velocity of S_2 relative to S_1 is v_1, that of S_3 relative to S_2 is v_2. A particle moves parallely as well with velocity v with respect to S_1. Find its velocity, say u, with respect to S_3.

Exercise 10. * Prove the following theorem. Any restricted Lorentz transformation is uniquely given as the product of a boost and a rotation:

$$\Lambda = L(\boldsymbol{v})\mathcal{R} =: L(\boldsymbol{v}) \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix},$$

with $R \in SO(3)$. Moreover, $v^i = \Lambda^i{}_0/\Lambda^0{}_0$, and

$$R^{ik} = \Lambda^i{}_k - \frac{\Lambda^i{}_0\Lambda^0{}_k}{1 + \Lambda^0{}_0} \tag{1.8}$$

Exercise 11. Prove the following assertions. It is always possible to find a frame in which a timelike vector T has the form $(x^0, \mathbf{0})$. The sign of x^0 is invariant under $L_+^\uparrow$. A (nonzero) vector orthogonal to t is spacelike. It is always possible to find a frame in which a spacelike vector s has the form $(0, \boldsymbol{s})$. A vector orthogonal to s is in general a superposition of a spacelike and a timelike vector. It is always possible to find a frame in which a null vector l has the form $(k, 0, 0, k)$. The sign of k is invariant under $L_+^\uparrow$. A vector orthogonal to l is a superposition of a spacelike vector and l itself.

1.3 Some group theory: generalities

The student of this course is assumed familiar with the concepts of Lie group and Lie algebra theory. It is clear that the infinitesimal generators of the boosts are

$$K^1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad K^2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad K^3 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We can thus rewrite every $\Lambda \in L_+^\uparrow$ as

$$\exp(\zeta \boldsymbol{n} \cdot \boldsymbol{K}) \exp(\alpha \boldsymbol{m} \cdot \boldsymbol{J}),$$

and

$$J^1 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \; J^2 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}; \; J^3 := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

for the infinitesimal generators of rotations. It is then easy to see that

$$[J^1, K^1] = 0; \quad [J^1, K^2] = K_3; \quad [K^1, K^2] = -J^3,$$

plus cyclic permutations, constitute the commutation relations for the Lorentz group.

We present a more covariant-looking form. If

$$x^{\alpha\prime} \simeq x^\alpha + \lambda^\alpha_{\ \nu} x^\nu$$

is an infinitesimal Lorentz transformation, then $\lambda_{\alpha\nu} := g_{\alpha\kappa}\lambda^\kappa_{\ \nu} = -\lambda_{\nu\alpha}$. Identifying the transformation parametres with the $\lambda_{\alpha\nu}$, one can write

$$x^{\alpha\prime} \simeq x^\alpha + \tfrac{1}{2}\lambda_{\kappa\rho}(M^{\kappa\rho})^\alpha_{\ \nu} x^\nu,$$

with the Lorentz group generators given by

$$(M^{\kappa\rho})^\mu_{\ \nu} = g^{\kappa\mu} g^\rho_\nu - g^{\rho\mu} g^\kappa_\nu = -(M^{\rho\kappa})^\mu_{\ \nu}. \tag{1.9}$$

In particular

$$K^i = M^{0i}, \qquad J^i = \varepsilon^i_{jk} M^{jk}. \tag{1.10}$$

The commutation relations are

$$[M^{\kappa\rho}, M^{\mu\nu}] = g^{\kappa\mu} M^{\rho\nu} + g^{\rho\nu} M^{\kappa\mu} - g^{\kappa\nu} M^{\rho\mu} - g^{\rho\mu} M^{\kappa\nu}.$$

In any given representation, the generators will be given by formulae different from (1.9), which is valid for the vector representation; but the abstract relations (1.10) will be kept, as well as the integrated formula

$$U(\lambda) = \exp\Bigl(\tfrac{1}{2}\lambda_{\kappa\rho}M^{\kappa\rho}\Bigr),$$

with the $M^{\kappa\rho}$ in the appropriate form.

If $L_1 \simeq 1 + \zeta_1 K_1 + \frac{1}{2}\zeta_1^2 K_1^2$, $L_2 \simeq 1 + \zeta_2 K_2 + \frac{1}{2}\zeta_2^2 K_2^2$, then for the group commutator:

$$L_2^{-1}L_1^{-1}L_2L_1 \simeq 1 - \zeta_1\zeta_2[K_1, K_2] = 1 + \zeta_1\zeta_2 J_3;$$

this is the (infinitesimal) Wigner rotation —of which more later. Globally, it is obtained as follows:

$$L(\boldsymbol{v}_1)L(\boldsymbol{v}_2) = \mathcal{R}(\boldsymbol{v}_1, \boldsymbol{v}_2)L(\boldsymbol{v}_1 \circ \boldsymbol{v}_2); \tag{1.11}$$

by definition $\mathcal{R}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ is the Wigner rotation and $\boldsymbol{v}_1 \circ \boldsymbol{v}_2$ is precisely the relativistic addition of velocities. It is $\mathcal{R}(\boldsymbol{v}, \boldsymbol{0}) = \mathcal{R}(-\boldsymbol{v}, \boldsymbol{v}) = \mathrm{Id}$.

It should be clear that although $\mathcal{R}L(\boldsymbol{v})\mathcal{R}^{-1} = L(\mathcal{R}\boldsymbol{v})$, the Lorentz group is *not* the semidirect product of a group of boosts by the group of rotations; actually the (restricted) Lorentz group is *simple*, that is, it is a noncommutative group without trivial invariant subgroups. A proof uses the facts that the rotation group is simple and that $L(\boldsymbol{u}) = L(\boldsymbol{v})L(\boldsymbol{v})$ is solvable for $\boldsymbol{v}$. We go not into that yet.

1.4 Poincaré adjoint and coadjoint actions

Quantum free particle (pure) states are the quantum elementary systems, given by the unirreps of $\mathcal{P}$. Our intuition of them is powerfully served by considering first *classical* elementary systems, which are orbits of the dual adjoint action of $\mathcal{P}$ on the linear dual of its Lie algebra.

Denote by $\mathcal{P}_0 = T_4 \ltimes \mathcal{L}_+^\uparrow = T_4 \ltimes SO_0(3,1)$, the proper orthochronous Poincaré group. We work with its simply connected double cover $\widetilde{\mathcal{P}}_0 := T_4 \ltimes SL(2,\mathbb{C})$. (This ensures that only linear representations need be considered.) The product on $\widetilde{\mathcal{P}}_0$ obeys

$$(a, A) \cdot (a', A') = (a + \Lambda_A a', AA') \quad \text{for} \quad a \in T_4,\ A \in SL(2,\mathbb{C}). \tag{1.12}$$

The Lie algebra $\mathfrak{p}$ of $\widetilde{\mathcal{P}}_0$ (or of $\mathcal{P}_0$ or $\mathcal{P}$) is generated by ten elements H, P^i, J^i, K^i (for $i = 1, 2, 3$) corresponding respectively to time translations, space translations, and the rotations and pure boosts of the Lorentz subgroup. We write elements of $\widetilde{\mathcal{P}}_0$ in a standard form

$$g = \exp(-a^0 H + \boldsymbol{a}\cdot\boldsymbol{P})\exp(\zeta\boldsymbol{n}\cdot\boldsymbol{K})\exp(\alpha\boldsymbol{m}\cdot\boldsymbol{J}),$$

where $a \in T_4$, $\boldsymbol{n}$ and $\boldsymbol{m}$ are unit 3-vectors, $\zeta \geq 0$ and $0 \leq \alpha \leq 2\pi$, with the understanding that $\exp(2\pi\boldsymbol{m}\cdot\boldsymbol{J}) = -1_2$ in $SL(2,\mathbb{C})$ for all $\boldsymbol{m}$. The full nonvanishing commutation relations for the generators:

$$\begin{aligned} &[J^i, J^j] = \varepsilon^{ij}{}_k J^k, && [J^i, K^j] = \varepsilon^{ij}{}_k K^k, && [J^i, P^j] = \varepsilon^{ij}{}_k P^k, \\ &[K^i, K^j] = -\varepsilon^{ij}{}_k J^k, && [K^i, P^j] = \delta^{ij} H, && [K^i, H] = P^i \end{aligned} \tag{1.13}$$

Table 1: The adjoint action $\mathrm{Ad}(\exp X)Y$

X/Y	$-a^0H$	$\boldsymbol{a}\cdot\boldsymbol{P}$	$\alpha\boldsymbol{m}\cdot\boldsymbol{J}$	$\zeta\boldsymbol{n}\cdot\boldsymbol{K}$
H	H	H	H	$(\cosh\zeta)H + (\sinh\zeta)\boldsymbol{n}\cdot\boldsymbol{P}$
$\boldsymbol{P}$	$\boldsymbol{P}$	$\boldsymbol{P}$	$R_{\alpha\boldsymbol{m}}^{-1}\boldsymbol{P}$	$\boldsymbol{P} + (\sinh\zeta)H\boldsymbol{n} + (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{P})\boldsymbol{n}$
$\boldsymbol{J}$	$\boldsymbol{J}$	$\boldsymbol{J} - \boldsymbol{a}\times\boldsymbol{P}$	$R_{\alpha\boldsymbol{m}}^{-1}\boldsymbol{J}$	$(\cosh\zeta)\boldsymbol{J} - (\sinh\zeta)\boldsymbol{n}\times\boldsymbol{K} - (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{J})\boldsymbol{n}$
$\boldsymbol{K}$	$\boldsymbol{K} + a^0\boldsymbol{P}$	$\boldsymbol{K} - H\boldsymbol{a}$	$R_{\alpha\boldsymbol{m}}^{-1}\boldsymbol{K}$	$(\cosh\zeta)\boldsymbol{K} + (\sinh\zeta)\boldsymbol{n}\times\boldsymbol{J} - (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{K})\boldsymbol{n}$

are obtained from $\boldsymbol{K} = \frac{1}{2}\boldsymbol{\sigma}$, $\boldsymbol{J} = -\frac{i}{2}\boldsymbol{\sigma}$. It ensues:

$$\exp(\zeta\boldsymbol{n}\cdot\boldsymbol{K}) = \cosh\frac{\zeta}{2} + \sinh\frac{\zeta}{2}\boldsymbol{n}\cdot\boldsymbol{\sigma} = \begin{pmatrix} \cosh\frac{\zeta}{2} + n_3\sinh\frac{\zeta}{2} & (n_1 - in_2)\sinh\frac{\zeta}{2} \\ (n_1 + in_2)\sinh\frac{\zeta}{2} & \cosh\frac{\zeta}{2} - n_3\sinh\frac{\zeta}{2} \end{pmatrix},$$
$$\exp(\alpha\boldsymbol{m}\cdot\boldsymbol{J}) = \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\boldsymbol{m}\cdot\boldsymbol{\sigma} = \begin{pmatrix} \cos\frac{\alpha}{2} - im_3\sin\frac{\alpha}{2} & (-im_1 - m_2)\sin\frac{\alpha}{2} \\ (-im_1 + m_2)\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} + im_3\sin\frac{\alpha}{2} \end{pmatrix}. \tag{1.14}$$

The adjoint action of $\widetilde{\mathcal{P}}_0$ on $\mathfrak{p}$ is computed as follows. Writing $\mathrm{ad}(X)Y := [X, Y]$ for $X, Y \in \mathfrak{p}$, we have $\mathrm{Ad}(\exp X)Y = e^{\mathrm{ad}(X)}Y = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \cdots$. From this it is easy to find $\mathrm{Ad}(\exp X)Y$ whenever $X = -a^0H$, $\boldsymbol{a}\cdot\boldsymbol{P}$, $\alpha\boldsymbol{m}\cdot\boldsymbol{J}$ or $\zeta\boldsymbol{n}\cdot\boldsymbol{K}$, and $Y = H$, P^i, J^i or K^i. For instance, if $X = \zeta\boldsymbol{n}\cdot\boldsymbol{K}$, $Y = H$, then

$$\begin{aligned}
&\mathrm{Ad}(\exp(\zeta\boldsymbol{n}\cdot\boldsymbol{K}))H \\
&\quad = H + \zeta[\boldsymbol{n}\cdot\boldsymbol{K}, H] + \frac{\zeta^2}{2!}[\boldsymbol{n}\cdot\boldsymbol{K}, [\boldsymbol{n}\cdot\boldsymbol{K}, H]] + \frac{\zeta^3}{3!}[\boldsymbol{n}\cdot\boldsymbol{K}, [\boldsymbol{n}\cdot\boldsymbol{K}, [\boldsymbol{n}\cdot\boldsymbol{K}, H]]] + \cdots \\
&\quad = H + \zeta\boldsymbol{n}\cdot\boldsymbol{P} + \frac{\zeta^2}{2!}H + \frac{\zeta^3}{3!}\boldsymbol{n}\cdot\boldsymbol{P} + \cdots = (\cosh\zeta)H + (\sinh\zeta)\boldsymbol{n}\cdot\boldsymbol{P}.
\end{aligned}$$

In this way one obtains Table 1, exhibiting the adjoint action of $\widetilde{\mathcal{P}}_0$ in a fully explicit manner [1]. (We write $\Lambda_A = R_{\alpha\boldsymbol{m}}$ for the rotation obtained from $A = \exp(\alpha\boldsymbol{m}\cdot\boldsymbol{J}) \in SU(2)$.)

The coadjoint action of $\widetilde{\mathcal{P}}_0$ on the Lie coalgebra $\mathfrak{p}^*$ is he contragredient of the adjoint representation, namely, if $\langle u, X\rangle := u(X)$ for $u \in \mathfrak{p}^*, X \in \mathfrak{p}$, then

$$\langle \mathrm{Coad}(g)u, X\rangle := \langle u, \mathrm{Ad}(g^{-1})X\rangle.$$

It can now be derived immediately. Let h be the linear coordinate on $\mathfrak{p}^*$ associated to H, and similarly let p^i, j^i, k^i be the coordinates associated to P^i, J^i, K^i $(i = 1, 2, 3)$. The action is given in these coordinates by Table 2.

Exercise 12. Verify formulas (1.14), using $(\sigma y)(\sigma z) = (\sigma w)$, where $w = (y^0z^0 + \boldsymbol{y}\cdot\boldsymbol{z}, y^0\boldsymbol{z} + z^0\boldsymbol{y} + i\boldsymbol{y}\wedge\boldsymbol{z})$.

Exercise 13. Consider $X_{(h,\boldsymbol{p})} = h1_2 + \boldsymbol{p}\cdot\boldsymbol{\sigma}$. Note that the coadjoint action is given simply by

$$X_{(h,\boldsymbol{p})} \mapsto A^{\dagger^{-1}}X_{(h,\boldsymbol{p})}A^{-1}, \tag{1.15}$$

for any $g = (a, A)$. For instance, prove the following.

$$\begin{aligned}
&\begin{pmatrix} \cos\frac{\alpha}{2} - im_3\sin\frac{\alpha}{2} & (-im_1 - m_2)\sin\frac{\alpha}{2} \\ (-im_1 + m_2)\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} + im_3\sin\frac{\alpha}{2} \end{pmatrix} X_{(h,\boldsymbol{p})} \begin{pmatrix} \cos\frac{\alpha}{2} + im_3\sin\frac{\alpha}{2} & (im_1 + m_2)\sin\frac{\alpha}{2} \\ (im_1 - m_2)\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} - im_3\sin\frac{\alpha}{2} \end{pmatrix} \\
&\quad = h1_2 + \big((\cos\alpha)\boldsymbol{p} + (\sin\alpha)\boldsymbol{m}\times\boldsymbol{p} + (1 - \cos\alpha)(\boldsymbol{m}\cdot\boldsymbol{p})\boldsymbol{m}\big)\cdot\boldsymbol{\sigma} = X_{(h,R_{\alpha\boldsymbol{m}}\boldsymbol{p})};
\end{aligned}$$

Table 2: The coadjoint action $\mathrm{Coad}(\exp X)y$

X/y	$-a^0 H$	$\boldsymbol{a}\cdot\boldsymbol{P}$	$\alpha\boldsymbol{m}\cdot\boldsymbol{J}$	$\zeta\boldsymbol{n}\cdot\boldsymbol{K}$
h	h	h	h	$(\cosh\zeta)h - (\sinh\zeta)\boldsymbol{n}\cdot\boldsymbol{p}$
$\boldsymbol{p}$	$\boldsymbol{p}$	$\boldsymbol{p}$	$R_{\alpha\boldsymbol{m}}\,\boldsymbol{p}$	$\boldsymbol{p} - (\sinh\zeta)h\boldsymbol{n} + (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n}$
$\boldsymbol{j}$	$\boldsymbol{j}$	$\boldsymbol{j} + \boldsymbol{a}\times\boldsymbol{p}$	$R_{\alpha\boldsymbol{m}}\,\boldsymbol{j}$	$(\cosh\zeta)\boldsymbol{j} + (\sinh\zeta)\boldsymbol{n}\times\boldsymbol{k} - (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{j})\boldsymbol{n}$
$\boldsymbol{k}$	$\boldsymbol{k} - a^0\boldsymbol{p}$	$\boldsymbol{k} + h\boldsymbol{a}$	$R_{\alpha\boldsymbol{m}}\,\boldsymbol{k}$	$(\cosh\zeta)\boldsymbol{k} - (\sinh\zeta)\boldsymbol{n}\times\boldsymbol{j} - (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{k})\boldsymbol{n}$

as well as

$$\begin{aligned}
&\begin{pmatrix} \cosh\frac{\zeta}{2} - n_3\sinh\frac{\zeta}{2} & -(n_1 - in_2)\sinh\frac{\zeta}{2} \\ -(n_1 + in_2)\sinh\frac{\zeta}{2} & \cosh\frac{\zeta}{2} + n_3\sinh\frac{\zeta}{2} \end{pmatrix} X_{(h,\boldsymbol{p})} \begin{pmatrix} \cosh\frac{\zeta}{2} - n_3\sinh\frac{\zeta}{2} & -(n_1 - in_2)\sinh\frac{\zeta}{2} \\ -(n_1 + in_2)\sinh\frac{\zeta}{2} & \cosh\frac{\zeta}{2} + n_3\sinh\frac{\zeta}{2} \end{pmatrix} \\
&= \big((\cosh\zeta)h - (\sinh\zeta)\boldsymbol{n}\cdot\boldsymbol{p}\big)1_2 + \big(\boldsymbol{p} - (\sinh\zeta)h\boldsymbol{n} + (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n}\big)\cdot\boldsymbol{\sigma} = X_{e^{\zeta\boldsymbol{n}\cdot\boldsymbol{K}}(h,\boldsymbol{p})}.
\end{aligned}$$

(It is natural that if A acts on configuration space by $\Lambda_A x$, it act contragrediently by $\Lambda_{A^{\dagger -1}}$ on momentum.)

2 Theory of free one-particle states*

2.1 Classical relativistic elementary systems

Classical elementary systems are orbits of the coadjoint action of a dynamical group on the linear dual of its Lie algebra. Those orbits arise in the present context as level sets of two “Casimir functions” C_1, C_2 on $\mathfrak{p}^*$. We obtain them explicitly. Let $p = (h, \boldsymbol{p})$ be the “energy-momentum” 4-vector and $w = (w^0, \boldsymbol{w})$ the “Pauli–Lubański” 4-vector, given by

$$w^0 = \boldsymbol{j}\cdot\boldsymbol{p}; \qquad \boldsymbol{w} = \boldsymbol{p}\times\boldsymbol{k} + h\boldsymbol{j}. \tag{2.1}$$

From Table 2, one verifies that w^0 transforms like h and $\boldsymbol{w}$ like $\boldsymbol{p}$ under the coadjoint action; in particular, under $\mathrm{Coad}\big(\exp(\zeta\boldsymbol{n}\cdot\boldsymbol{K})\big)$:

$$\begin{aligned}
w^0 &\mapsto (\cosh\zeta)w^0 - (\sinh\zeta)\boldsymbol{n}\cdot\boldsymbol{w}, \\
\boldsymbol{w} &\mapsto \boldsymbol{w} - (\sinh\zeta)w^0\boldsymbol{n} + (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{w})\boldsymbol{n}.
\end{aligned}$$

It is easily checked that p and w are orthogonal in the Minkowski sense: $(pw) = 0$. Thus the Casimir functions we seek are

$$C_1 := (pp) = h^2 - |\boldsymbol{p}|^2, \qquad C_2 := (ww) = (\boldsymbol{j}\cdot\boldsymbol{p})^2 - |\boldsymbol{p}\times\boldsymbol{k} + h\boldsymbol{j}|^2.$$

We now look for orbits corresponding to physical particles, starting by the massive case. So we restrict ourselves to orbits for which $C_1 < 0$, writing $C_1 = m^2$ with $m > 0$, and $h = \sqrt{m^2 + |\boldsymbol{p}|^2} > 0$. Let $\kappa := (m, \boldsymbol{0})$ be the vertex of the forward hyperboloid $p^2 = m^2$, $p^0 > 0$. Consider the Lorentz boost L_u which takes p to κ or $u := p/m$ to $(1, 0)$. We have $0 = (pw) = (L_u p\, L_u w) = (\kappa\, L_u w)$, which means that $L_u w = (0, m\boldsymbol{s})$ for some 3-vector $\boldsymbol{s}$. Since

$C_2 = (ww) = (L_u w\, L_u w) = m^2|\boldsymbol{s}|^2$ is constant on any orbit, one interprets $\boldsymbol{s}$ as the **spin vector** —which thus is intrinsically spacelike. From $(0, m\boldsymbol{s}) = L_u w$, it yields:

$$\boldsymbol{s} = \frac{\boldsymbol{w}}{m} - \frac{1}{m}\left(\frac{w^0}{m} - \frac{\boldsymbol{p}\cdot\boldsymbol{w}}{m(h+m)}\right)\boldsymbol{p} = \frac{\boldsymbol{w}}{m} - \frac{w^0\boldsymbol{p}}{m(h+m)}. \tag{2.2}$$

For fixed m and s and positive h, we obtain a single orbit $\mathcal{O}_{ms+}$. If $s > 0$, we may take as coordinates on $\mathcal{O}_{ms+}$ the momenta $\boldsymbol{p}$ and spherical coordinates arising from $\boldsymbol{s}$; three coordinates remain to be determined. A possible choice is $\boldsymbol{q}$, given by

$$\boldsymbol{q} := \frac{\boldsymbol{k}}{h} - \frac{\boldsymbol{p}\times\boldsymbol{w}}{mh(m+h)} = \frac{\boldsymbol{k}}{h} - \frac{\boldsymbol{p}\times\boldsymbol{s}}{h(m+h)}. \tag{2.3}$$

So the coadjoint orbit $\mathcal{O}_{ms+}$ is homeomorphic to $\mathbb{R}^6 \times \mathbb{S}^2$ —with "little group" $\mathbb{R} \times SO(2)$. By general theory, the *Poisson bracket* on the codajoint orbit is given by

$$\{f, g\} = c^k_{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}x_k, \tag{2.4}$$

where c^k_{ij} denote the structure constants of the Lie algebra. Using this together with (1.13), one can check that $\{\, q^i, p^i \,\}$ are (part of a set of) *canonical* coordinates. It follows that $d^3q\, d^3p\, d\boldsymbol{s}$ is a Liouville measure on $\mathcal{O}_{ms+}$. The case $s = 0$ gives a 6-dimensional orbit $\mathcal{O}_{m0+}$, isomorphic to $\mathbb{R}^6$. From (2.1) the expressions of the $\mathfrak{p}$ coordinates $(h, \boldsymbol{p}, \boldsymbol{j}, \boldsymbol{k})$ in terms of the $\mathcal{O}_{ms+}$ coordinates $(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{s})$ over the orbit are:

$$\begin{aligned} w^0 &= \boldsymbol{p}\cdot\boldsymbol{s}, \qquad & \boldsymbol{w} &= m\boldsymbol{s} + \frac{\boldsymbol{p}\cdot\boldsymbol{s}}{m+h}\boldsymbol{p}, \\ \boldsymbol{j} &= \boldsymbol{q}\times\boldsymbol{p} + \boldsymbol{s}, \qquad & \boldsymbol{k} &= h\boldsymbol{q} + \frac{\boldsymbol{p}\times\boldsymbol{s}}{m+h}. \end{aligned} \tag{2.5}$$

Now we can recover from Table 2 the expression of the coadjoint action of $\widetilde{\mathcal{P}}_0$ on the orbit in terms of the coordinates $(\boldsymbol{p}, \boldsymbol{q})$. There is no need to rewrite the action on $\boldsymbol{p}$. We readily obtain:

$$\begin{aligned} \exp(-a^0 H) \triangleright \boldsymbol{q} &= \boldsymbol{q} - \frac{a_0\boldsymbol{p}}{h} \\ \exp(\boldsymbol{a}\cdot\boldsymbol{P}) \triangleright \boldsymbol{q} &= \boldsymbol{q} + \boldsymbol{a} \\ \exp(\alpha\boldsymbol{m}\cdot\boldsymbol{J}) \triangleright \boldsymbol{q} &= R_{\alpha\boldsymbol{m}}\boldsymbol{q}. \end{aligned} \tag{2.6}$$

These Euclidean transformation rules conform to our intuition as to how a relativistic particle should behave. Nevertheless, contrary to the momentum representation, the issue of position variables and position representation is a complicated one in relativistic quantum mechanics. For instance, $\boldsymbol{q}$ does not behave covariantly under boosts when $s > 0$.

Exercise 14. Show that expression (2.1) is given in covariant form by $W_\sigma = \frac{1}{2}\varepsilon_{\sigma\mu\nu\lambda}M^{\mu\nu}P^\lambda$, with $\varepsilon_{0123} = -1$.

Exercise 15. Write L_u explicitly and show that its relation to Coad$\big(\exp(\zeta\boldsymbol{n}\cdot\boldsymbol{K})\big)$ is given by

$$\boldsymbol{n} = \frac{-\boldsymbol{u}}{|\boldsymbol{u}|} = \frac{-\boldsymbol{u}}{\sqrt{(u^0)^2 - 1}}; \quad \zeta = \cosh^{-1}(u^0).$$

Exercise 16. Prove

$$\{\, s^i, s^j \,\} = \varepsilon^{ij}_k s^k,$$

from the general formula (2.4).

2.1.1 Position coordinates in the massive case

Besides $\boldsymbol{q}$ there are other interesting position variables. For instance, for massive particles one can always find a *covariant* coordinate vector, fulfilling like $\boldsymbol{q}$ Euclidean transformation rules, but coinciding with the canonical coordinate vector only for the spinless case. This is of the form:

$$\boldsymbol{x} = \frac{\boldsymbol{k}}{h} - \frac{\boldsymbol{w}\times\boldsymbol{s}}{m^2h} = \frac{\boldsymbol{k}}{h} - \frac{\boldsymbol{p}\times\boldsymbol{s}}{mh} = \boldsymbol{q} - \frac{\boldsymbol{p}\times\boldsymbol{s}}{m(m+h)}.$$

Covariance means the rule of transformation of the *initial coordinates* for free motion on changing from one Lorentz frame to another given rise the relativistic transformation rule for such motion. That is to say, if $\boldsymbol{x}(t) = \boldsymbol{x} + \boldsymbol{p}t/h$; $\boldsymbol{x}'(t') = \boldsymbol{x}' + \boldsymbol{p}'t'/h'$, together with $t' = t\cosh\zeta - (\sinh\zeta)\boldsymbol{x}(t)\cdot\boldsymbol{n}$, then one can show indeed

$$\boldsymbol{x}'(t') = \boldsymbol{x}(t) - t(\sinh\zeta)\boldsymbol{n} + (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{x}(t))\boldsymbol{n}. \tag{2.7}$$

We shall soon see that for massless particles (with non-vanishing helicity) we can have neither canonical nor covariant coordinates. It is therefore astute to seek already a position vector with *good limit* properties as $m \downarrow 0$. Such coordinates $\boldsymbol{X}$ are provided by

$$\boldsymbol{X} := \frac{\boldsymbol{k}}{h} + \frac{\boldsymbol{p}\times\boldsymbol{w}}{h^2(m+h)} = \boldsymbol{q} + \frac{\boldsymbol{p}\times\boldsymbol{s}}{h^2}.$$

In terms of the $(\boldsymbol{X}, \boldsymbol{p}, \boldsymbol{s})$ coordinates, we have now:

$$\boldsymbol{j} = \boldsymbol{X}\times\boldsymbol{p} + \boldsymbol{s}\frac{m^2}{h^2} + \boldsymbol{p}\frac{\boldsymbol{p}\cdot\boldsymbol{s}}{h^2}; \qquad \boldsymbol{k} = h\boldsymbol{X} - m\frac{\boldsymbol{p}\times\boldsymbol{s}}{h(m+h)}. \tag{2.8}$$

Exercise 17. Compute the Poisson brackets $\{X^i, X^j\}$.

Exercise 18. Prove that (2.6) holds for $\boldsymbol{X}$.

2.1.2 Wigner rotations

Next we describe the coadjoint action on spin. Unsurpringly, we have:

$$\begin{aligned} &\exp(-a^0 H)\triangleright\boldsymbol{s} = \boldsymbol{s}, \quad \exp(\boldsymbol{a}\cdot\boldsymbol{P})\triangleright\boldsymbol{s} = \boldsymbol{s}, \\ &\exp(\alpha\boldsymbol{m}\cdot\boldsymbol{J})\triangleright\boldsymbol{s} = R_{\alpha\boldsymbol{m}}\boldsymbol{s}. \end{aligned} \tag{2.9}$$

The action under boosts $B\triangleright\boldsymbol{s} \equiv \exp(\zeta\boldsymbol{n}\cdot\boldsymbol{K})\triangleright\boldsymbol{s}$ is more complicated. Indeed, if $w' = Bw$, then $(0, m\boldsymbol{s}') = L_{Bu}w' = L_{Bu}Bw = L_{Bu}BL_u^{-1}(0, m\boldsymbol{s})$. So we have

$$B\triangleright\boldsymbol{s} = L_{Bu}BL_u^{-1}\boldsymbol{s} = R_W(B,u)\boldsymbol{s}.$$

The Lorentz group element $R_W(B,u)$ must be the a rotation: to wit, the famous Wigner (Thomas) rotation corresponding to B and u. In view of (2.9) this generalizes in the obvious way to any Lorentz transformation at the place of B. See [2] for a timely reminder on Wigner rotations. Note the kinship with definition (1.11).

A detailed discussion of R_W serves our purposes. The spin's axis of rotation is given by $\boldsymbol{p}\times\boldsymbol{n}$, if B is a boost in the direction of $\boldsymbol{n}$: when the boost B is parallel to the momentum $\boldsymbol{p}$, there is no Wigner rotation. With $\boldsymbol{m} = \frac{\boldsymbol{p}\times\boldsymbol{n}}{|\boldsymbol{p}\times\boldsymbol{n}|}$, one has [1]:

$$\boldsymbol{s}' = R_W\boldsymbol{s} = R_{\delta\boldsymbol{m}}\boldsymbol{s} = \cos\delta\,\boldsymbol{s} + \sin\delta\,\boldsymbol{m}\times\boldsymbol{s} + (1-\cos\delta)(\boldsymbol{m}\cdot\boldsymbol{s})\boldsymbol{m},$$

$$\text{where} \quad \sin\delta = \frac{\sin\zeta(m+h) - (\cosh\zeta - 1)(\boldsymbol{p}\cdot\boldsymbol{n})}{(m+h)(m+h')}|\boldsymbol{p}\times\boldsymbol{n}|.$$

A key point is that (although not all the factors in its definition) the Wigner rotation formula makes perfect sense for $m = 0$, namely:

$$\sin\delta = \frac{h\,\sin\zeta - (\cosh\zeta - 1)(\boldsymbol{p}\cdot\boldsymbol{n})}{hh'}|\boldsymbol{p}\times\boldsymbol{n}|, \tag{2.10}$$

keeping in mind that in this case $h = |\boldsymbol{p}|, h' = |\boldsymbol{p}'|$. Meanwhile the momentum under B also turns around $\boldsymbol{p}\times\boldsymbol{n}$. This is true in all generality: from the coadjoint action

$$\boldsymbol{p}' = \boldsymbol{p} - (\sinh\zeta)h\boldsymbol{n} + (\cosh\zeta - 1)(\boldsymbol{n}\cdot\boldsymbol{p})\boldsymbol{n}$$

we have

$$\boldsymbol{p}'\times\boldsymbol{p} = \big[h\sinh\zeta - (\cosh\zeta - 1)(\boldsymbol{p}\cdot\boldsymbol{n})\big]\boldsymbol{p}\times\boldsymbol{n};$$

therefore the component of $\boldsymbol{p}'$ not along $\boldsymbol{p}$ stays on the plane perpendicular to $\boldsymbol{p}\times\boldsymbol{n}$. Now, we compute the angle:

$$\frac{|\boldsymbol{p}'\times\boldsymbol{p}|}{|\boldsymbol{p}||\boldsymbol{p}'|} = \frac{h\sinh\zeta - (\cosh\zeta - 1)(\boldsymbol{p}\cdot\boldsymbol{n})}{|\boldsymbol{p}||\boldsymbol{p}'|}|\boldsymbol{p}\times\boldsymbol{n}|.$$

This is in general bigger than the Wigner angle; but clearly in the massless limit —so $h = |\boldsymbol{p}|$— momentum and spin turn in perfect solidarity.

2.1.3 The massless case

We next look for the orbits corresponding to massless particles, determined by $C_1 = 0$. Clearly $\boldsymbol{p}\in\mathbb{R}^3\setminus 0$ (the origin is an orbit). We consider the case $h > 0$ and make the critical assumption that w is parallel to p, that is $w = \lambda p$ with $\lambda\in\mathbb{R}$. Taking the limit as $m\downarrow 0$ in (2.8), also on account of (2.1), everything is determined:

$$\boldsymbol{p} = \boldsymbol{p}, \qquad |\boldsymbol{p}| = h, \qquad \boldsymbol{j} = \boldsymbol{X}\times\boldsymbol{p} + \lambda\frac{\boldsymbol{p}}{h}, \qquad \boldsymbol{k} = h\boldsymbol{X}. \tag{2.11}$$

Here the **helicity** $\lambda = \boldsymbol{j}\cdot\boldsymbol{p}/h$ is the projection of the total angular momentum $\boldsymbol{j}$ on the momentum; this is what remains of the spin in the massless case. The orbit $\mathcal{O}_\lambda$ is therefore six-dimensional, and isomorphic to $\mathbb{R}^3\times(\mathbb{R}^3\setminus 0)\simeq\mathbb{R}^3\times\mathbb{R}\times\mathbb{S}^2$. This non-trivial topology has some non-trivial consequences. Observe that we have got rid of expressions in terms of $\boldsymbol{s}$: it ought to be possible to do so, since the orbit is only six-dimensional.

Hence, in view of (1.13):

$$\{p^i, X^j\} = \{p^i, h^{-1}k^j\} = h^{-1}\{p^i, k^j\} = -\delta_{ij}. \tag{2.12}$$

On the other hand,

$$\begin{aligned}\{X^i, X^j\} &= \{h^{-1}k^i, h^{-1}k^j\} = h^{-2}\{k^i, k^j\} + h^{-1}k^j\{k^i, h^{-1}\} + h^{-1}k^i\{h^{-1}, k^j\}\\ &= h^{-2}(-\varepsilon^{ij}{}_k j^k - X^j p^i + X^i p^j) = -\lambda\frac{\varepsilon^{ij}{}_k p^k}{h^3};\end{aligned} \tag{2.13}$$

so these are *not* canonical coordinates, unless $\lambda = 0$.

There are other coadjoint orbits, seemingly not correspondig to physical objects.

Exercise 19. Show by direct computation that the Poisson brackets of the helicity with all the other phase space variables vanish.

2.2 The massive unirreps of $\mathcal{P}$

In quantum mechanics the basis vectors of a physical Hilbert space $\mathcal{H}$ are always labeled by the (maybe generalized) eigenvalues of (a complete set of) commuting observables. Different observables mean different sets of basis vectors, related by unitary transformations (or "Clebsch–Gordan" expansions). Particularly important in particle physics are plane wave (generalized) eigenstates of momentum and the angular momentum states (of sundry kinds). The explicit form of those transformations depends on phase conventions, so the utmost care ought to be exercised in comparing extant results in the literature.

The classification of quantum 1-particle states is one and the same thing that the theory of unirreps of the Poincaré group. These are denoted by $U(a, A)$ with $U(0, A) = \exp(-i\boldsymbol{\alpha}\cdot\mathbb{J})$ and $U(0, A) = \exp(-i\zeta\cdot\mathbb{K})$ respectively for rotations and boosts, as well as $U(a^0, 1) = \exp(-ia^0\mathbb{H})$ and $U(\boldsymbol{a}, 1) = \exp(i\boldsymbol{a}\cdot\mathbb{P})$, where the infinitesimal operators $\mathbb{J}$, $\mathbb{K}$, $\mathbb{H}$, $\mathbb{P}$ are now self-adjoint operators on $\mathcal{H}$. In covariant notation,

$$U(a, 1) = e^{-i(\mathbb{P}a)}; \qquad U(0, A) = e^{-i(\alpha\mathbb{M})} := \exp\bigl(-i\alpha_{\mu\nu}\mathbb{M}^{\mu\nu}\bigr).$$

One has now the commutation relations:

$$\begin{aligned}
&[\mathbb{J}^i, \mathbb{J}^j] = i\varepsilon^{ij}{}_k\mathbb{J}^k, && [\mathbb{J}^i, \mathbb{K}^j] = i\varepsilon^{ij}{}_k\mathbb{K}^k, && [\mathbb{J}^i, \mathbb{P}^j] = i\varepsilon^{ij}{}_k\mathbb{P}^k, \\
&[\mathbb{K}^i, \mathbb{K}^j] = -i\varepsilon^{ij}{}_k\mathbb{J}^k, && [\mathbb{K}^i, \mathbb{P}^j] = i\delta^{ij}\mathbb{H}, && [\mathbb{K}^i, \mathbb{H}] = i\mathbb{P}^i.
\end{aligned}$$

The unirreps are characterized by the values of two invariants or *Casimir operators* which are the exact analogues of the classical entities studied in the first part of this section: $(\mathbb{PP})$ and $(\mathbb{WW})$, related to the rest mass and spin. The latter is invoked as follows. Analogously to (1.10) we introduce

$$\mathbb{M}^{0i} := \mathbb{K}^i, \qquad \mathbb{J}^i = \varepsilon^i_{jk}\mathbb{M}^{jk}, \quad \text{and} \quad \mathbb{W}_\sigma = \tfrac{1}{2}\varepsilon_{\sigma\mu\nu\lambda}\mathbb{M}^{\mu\nu}\mathbb{P}^\lambda.$$

Then one finds

$$[\mathbb{P}^\mu, \mathbb{W}_\sigma] = 0; \quad [\mathbb{M}_{\mu\nu}, \mathbb{W}_\sigma] = i(\mathbb{W}_\mu g_{\nu\sigma} - \mathbb{W}_\nu g_{\mu\sigma}); \quad [\mathbb{W}_\lambda, \mathbb{W}_\sigma] = i\varepsilon_{\lambda\sigma\mu\nu}\mathbb{W}^\mu\mathbb{P}^\nu. \tag{2.14}$$

Finally, one checks that $(\mathbb{WW})$ commutes with all the generators. The spin operator, with the well-known commutattion relations and eigenvalues, is just $\mathbb{W}$ in the rest system; formula (2.2) applies *mutatis mutandis*. As before, we consider first

$$(\mathbb{PP}) = m^2 > 0; \qquad (\mathbb{WW}) = -m^2\, s(s+1).$$

Given a state of a particle, irreducibility implies that any other possible state is obtained by means of Poincaré transformations acting on the original state vector. Thus consider a massive particle with spin at rest. Here $(\mathbb{PP})$, $(\mathbb{WW})$, $\mathbb{P}$, $\mathbb{J}_3$ form a complete set of quantum observables. There are $2s+1$ independent states corresponding to the third component of the angular momentum operator, so we have the (improper) kets $|\boldsymbol{\eta} = \mathbf{0}, r\rangle_{[m,j]} = |\kappa, r\rangle$, with $r = -j, -j+1, \dots, j$, spanning a $2j+1$-dimensional unirrep of $SU(2)$. Often the subscript $[m.j]$ identifying the unirreps will be omitted. We suppose the reader acquainted with the theory of angular momentum in quantum mechanics and its notations. In particular, we admit

$$\exp(-i\boldsymbol{\alpha}\cdot\mathbb{J})|\boldsymbol{\eta} = \mathbf{0}, r\rangle = \sum_{r'=-j}^{j} D^{(j)}_{r'r}(\boldsymbol{\alpha})|\boldsymbol{\eta} = \mathbf{0}, r'\rangle.$$

We proceed to define now states in motion. Let φ, ϑ be the polar angles of $\boldsymbol{\eta}$ and consider the rotation

$$R(\varphi,\vartheta,0)=\exp(-i\mathbb{J}^3\varphi)\exp(-i\mathbb{J}^2\vartheta),$$

taking the z-axis to the direction of $\boldsymbol{\eta}$, as well as the boost $\exp(-i\mathbb{K}^3 v)$, with $v=\tanh\bigl(|\boldsymbol{\eta}|/\eta^0\bigr)$. We may for instance introduce the following two states:

$$\begin{aligned}|\boldsymbol{\eta},r\rangle &:= R(\varphi,\vartheta,0)\exp(-i\mathbb{K}^3 v)R^{-1}(\varphi,\vartheta,0)|\boldsymbol{\eta}=\mathbf{0},r\rangle;\\ |\boldsymbol{\eta},\lambda\rangle &:= R(\varphi,\vartheta,0)\exp(-i\mathbb{K}^3 v)|\boldsymbol{\eta}=\mathbf{0},r\rangle.\end{aligned}\tag{2.15}$$

The last definition is not good for $\varphi=\pi$, whereupon one uses

$$|0,\pi,|\boldsymbol{\eta}|,\lambda\rangle=e^{-i\pi j}R(\pi,\pi,0)\exp(-i\mathbb{K}^3 v)|\boldsymbol{\eta}=\mathbf{0},r\rangle.$$

The second one is an helicity state, since it is an eigenvector of $\mathbb{J}\cdot\mathbb{P}/|\mathbb{P}|$. The vectors are normalized by

$$\langle\boldsymbol{\eta},r\mid\boldsymbol{\eta}',r'\rangle=2\eta^0\,\delta_{rr'}\delta(\boldsymbol{\eta}-\boldsymbol{\eta}');\quad \langle\boldsymbol{\eta},\lambda\mid\boldsymbol{\eta}',\lambda'\rangle=2\eta^0\,\delta_{\lambda\lambda'}\delta(\boldsymbol{\eta}-\boldsymbol{\eta}').\tag{2.16}$$

To show the consistency of the method, we perform a little calculation. We expect physically

$$\mathbb{P}^1|\boldsymbol{\eta},\lambda\rangle=|\boldsymbol{\eta}|\sin\vartheta\cos\varphi|\boldsymbol{\eta},\lambda\rangle.\tag{2.17}$$

Now we have

$$\begin{aligned}\mathbb{P}^1|\boldsymbol{\eta},\lambda\rangle &:= \mathbb{P}^1R(\varphi,\vartheta,0)\exp(-i\mathbb{K}^3v)|\boldsymbol{\eta}=\mathbf{0},r\rangle=\exp(-i\mathbb{J}^3\varphi)\exp(-i\mathbb{J}^2\vartheta)\exp(-i\mathbb{K}^3v)\\ &\quad\times\exp(i\mathbb{K}^3v)\exp(i\mathbb{J}^2\vartheta)\exp(i\mathbb{J}^3\varphi)\mathbb{P}^1\exp(-i\mathbb{J}^3\varphi)\exp(-i\mathbb{J}^2\vartheta)\exp(-i\mathbb{K}^3v)|\boldsymbol{\eta}=\mathbf{0},r\rangle\\ &\quad=\exp(-i\mathbb{J}^3\varphi)\exp(-i\mathbb{J}^2\vartheta)\exp(-i\mathbb{K}^3v)\exp(i\mathbb{K}^3v)\exp(i\mathbb{J}^2\vartheta)\bigl(\mathbb{P}^1\cos\varphi-\mathbb{P}^2\sin\varphi\bigr)\\ &\quad\times\exp(-i\mathbb{J}^2\vartheta)\exp(-i\mathbb{K}^3v)|\boldsymbol{\eta}=\mathbf{0},r\rangle=\exp(-i\mathbb{J}^3\varphi)\exp(-i\mathbb{J}^2\vartheta)\exp(-i\mathbb{K}^3v)\\ &\quad\times\bigl(\mathbb{P}^0\sin\vartheta\cos\varphi\sinh u+\mathbb{P}^1\cos\vartheta\cos\varphi-\mathbb{P}^2\sin\varphi+\mathbb{P}^3\sin\vartheta\cos\varphi\cosh u\bigr)|\boldsymbol{\eta}=\mathbf{0},r\rangle.\end{aligned}$$

The rest is easy.

Exercise 20. * Check that $\mathbb{W}^2=(\mathbb{W}\mathbb{W})$ is a Casimir.

Exercise 21. Verify equations (2.14).

Exercise 22. Finish the proof of (2.17).

For a massive particle, instead of (2.15) we may introduce (Wigner basis) states by

$$|\boldsymbol{\eta},r\rangle := L_{\eta/m}|\boldsymbol{\eta}=\mathbf{0},r\rangle;$$

it is then not hard to see that

$$U(\Lambda)|\boldsymbol{\eta},r\rangle_{[m,j]}=\sum_{r'=-j}^{j}D^{(j)}_{r'r}\bigl(R_W(\Lambda,\eta/m)\bigr)|\boldsymbol{\eta}=\mathbf{0},r'\rangle|\Lambda\boldsymbol{\eta},r'\rangle_{[m,j]},$$

where the Wigner rotation $R_W(\Lambda,\eta/m)$ slightly generalizes the previous case in which Λ was a boost.

We turn to the helicity basis (essential for massless particle states, studied later, and very useful for particle decay problems). Begin by a particle at rest, whose z-component of spin is λ, described by $|\mathbf{0},\lambda\rangle_{[m,j]}$. Given $\boldsymbol{\eta}$, we set that particle in motion in the z-direction with momentum of magnitude $|\boldsymbol{\eta}|$. We denote

$$||\boldsymbol{\eta}|\boldsymbol{e}_z,\lambda\rangle := U\big(L_{|\boldsymbol{\eta}|\boldsymbol{e}_z/m}\big)|\boldsymbol{\eta}=\mathbf{0},\lambda\rangle;$$

Next this state is *rotated* so that the momentum is $\boldsymbol{\eta}$; then the helicity does not change —it is a (pseudo)scalar. We set

$$|\boldsymbol{\eta},\lambda\rangle := U\big(R_{\boldsymbol{\eta},|\boldsymbol{\eta}|\boldsymbol{e}_z}\big)||\boldsymbol{\eta}|\boldsymbol{e}_z,\lambda\rangle = \sum_r|\boldsymbol{\eta},r\rangle D^{(j)}_{r\lambda}\big(L^{-1}_{\eta/m}R_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z}L_{|\boldsymbol{\eta}|\boldsymbol{e}_z/m}\big) \quad \text{and obtain}$$

$$\begin{aligned}
U(\Lambda)|\boldsymbol{\eta},\lambda\rangle &= \sum_r|\Lambda\boldsymbol{\eta},r\rangle D^{(j)}_{r\lambda}\big(L^{-1}_{\Lambda\eta/m}\Lambda R_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z}L_{|\boldsymbol{\eta}|\boldsymbol{e}_z/m}\big)\\
&= \sum_{rs}|\Lambda\boldsymbol{\eta},r\rangle D^{(j)}_{rs}\big(L^{-1}_{\Lambda\eta/m}R_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z}L_{|\boldsymbol{\eta}|\boldsymbol{e}_z/m}\big)D^{(j)}_{s\lambda}\big(L^{-1}_{|\boldsymbol{\eta}|\boldsymbol{e}_z/m}R^{-1}_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z}\Lambda R_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z}L_{|\boldsymbol{\eta}|\boldsymbol{e}_z/m}\big)\\
&=: \sum_\mu|\Lambda\boldsymbol{\eta},\mu\rangle D^{(j)}_{\mu\lambda}\big(L^{-1}_{|\boldsymbol{\eta}|\boldsymbol{e}_z/m}R^{-1}_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z}\Lambda R_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z}L_{|\boldsymbol{\eta}|\boldsymbol{e}_z/m}\big)\\
&= \sum_\mu|\Lambda\boldsymbol{\eta},\mu\rangle D^{(j)}_{\mu\lambda}\big(R^{-1}_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z}L^{-1}_{\Lambda\eta/m}\Lambda L_{\eta/m}R_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z|}\big).
\end{aligned}$$

In the last equality we have used that a boost in a given direction followed by a rotation can be replaced by the rotation and then the boost to the final momentum. In the end,

$$U(\Lambda)|\boldsymbol{\eta},\lambda\rangle = \sum_\mu|\Lambda\boldsymbol{\eta},\mu\rangle D^{(j)}_{\mu\lambda}\big(R^{-1}_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z}R_W(\Lambda,\eta/m)R_{\boldsymbol{\eta}|,\boldsymbol{\eta}|\boldsymbol{e}_z|}\big).$$

These relations are of interest to develop the relativistic version of angular momentum addition. A two-particle state given by the product of two states transforms like a reducible representation of the Poincaré group. It can be reduced to a sum of irreps, characterized by a mass $M^2 \equiv (\eta_1+\eta_2)^2$ and a total angular momentum made up by the addition of spins and orbital angular momentum, that serves as degeneracy parameter.

It is high time to come back to the description of the representation space and the concrete actions of the generators on it. From (2.16) we see that Wigner's canonical representation for massive relativistic particles lives on Hilbert spaces $\mathcal{H}_m$ spanned by the kets $|\boldsymbol{\eta},r\rangle_{[m,j]}$, with the vector $\boldsymbol{\eta}$ resting on the forward mass hyperboloid H^+_m and $r=-j,-j+1,\dots,j$, subject to the covariant closure and orthogonality relations

$$\begin{gathered}
\sum_r\int_{H^+_m}|\boldsymbol{\eta},r\rangle\langle\boldsymbol{\eta},r|\,\frac{d^3\eta}{2\eta^0} = I \quad \big(\eta^0=\sqrt{m^2+|\boldsymbol{\eta}|^2}\big);\\
\langle\boldsymbol{\eta},r\mid\boldsymbol{\eta}',r'\rangle = 2\eta^0\,\delta_{rr'}\delta(\boldsymbol{\eta}-\boldsymbol{\eta}').
\end{gathered} \tag{2.18}$$

The Weyl system generators $\mathbb{Q},\mathbb{P},\boldsymbol{S}$ are selfadjoint operators with non-vanishing commutation relations $[\mathbb{Q}^i,\mathbb{P}^j]=\delta_{ij}$, $[\boldsymbol{S}^i,\boldsymbol{S}^j]=i\varepsilon^{ij}_k\boldsymbol{S}^k$. In the chosen (momentum) representation they are given by

$$\mathbb{P}=\boldsymbol{\eta}; \qquad \mathbb{Q}=i\frac{\partial}{\partial\boldsymbol{\eta}}-i\frac{\boldsymbol{\eta}}{2|\boldsymbol{\eta}|^2};$$

plus the standard hermitian spin matrices vector. The other generators of the group are represented as follows:

$$\mathbb{H}=\eta^0:=\sqrt{m^2+\mathbb{P}^2}; \quad \mathbb{J}=\mathbb{Q}\times\mathbb{P}+\boldsymbol{S}; \quad \mathbb{K}=\tfrac{i}{2}[\mathbb{H},\mathbb{Q}]_+ + \frac{\mathbb{P}\times\boldsymbol{S}}{m+\mathbb{H}}. \tag{2.19}$$

Note $\mathbb{Q} = \frac{1}{2}[\mathbb{H}^{-1}, \mathbb{K}]_+$. Formula (2.18) says that the scalar product is explicitly given by

$$\langle \Psi \mid \Phi \rangle = \sum \int \bar{\Psi}(\boldsymbol{\eta})\Phi(\boldsymbol{\eta}) \, \frac{d^3\eta}{2\eta^0},$$

with sum over the $2j+1$ components understood. We may use instead use a "wave function" in configuration space,

$$\Psi(x) = (2\pi)^{-3/2} \int \Psi(\eta) e^{-i(\eta x)} \, d^4\eta = (2\pi)^{-3/2} \int \Psi(E_{\boldsymbol{\eta}}, \boldsymbol{\eta}) e^{i(\boldsymbol{\eta}\cdot\boldsymbol{x} - E_{\boldsymbol{\eta}} t)} \, \frac{d^3\eta}{2\eta^0}.$$

Then clearly the Klein–Gordon (KG) equation is satisfied by $\Psi(x)$:

$$(\Box + m^2)\Psi(x) = 0. \tag{2.20}$$

2.2.1 Massless representations

The identity $(\mathbb{P}^\mu \mathbb{W}_\mu) = 0$ is obvious. For a zero mass particle, there is no rest system, and it is inveterate custom to take as standard state Φ_0 the one characterized by momentum $(\eta^0, 0, 0, \eta^0)$. This can be reached by a simple rotation. We have then $\eta^0(\mathbb{W}_3 - \mathbb{W}_0)\Phi_0 = 0$, plus the commutation relations

$$[\mathbb{W}_1, \mathbb{W}_2]\Phi_0 = 0; \quad [\mathbb{W}_3, \mathbb{W}_1]\Phi_0 = i\eta^0 \mathbb{W}_2 \Phi_0; \quad [\mathbb{W}_3, \mathbb{W}_2]\Phi_0 = -i\eta^0 \mathbb{W}_1 \Phi_0.$$

These are the commutation relations of the Euclidean group, with $\mathbb{W}_3$ as the generator of rotations. We give to the Casimir $\mathbb{W}_1^2 + \mathbb{W}_2^2$ the value zero. Therefore, for all massless states,

$$\mathbb{W} \propto \mathbb{P}, \quad \text{so that} \quad \mathbb{W}_\mu = -\lambda \mathbb{P}_\mu,$$

where the constant λ is the helicity. Note that we have

$$\lambda = -\frac{(\mathbb{W}n)}{(\mathbb{P}n)},$$

for n an arbitrary vector. Taking $n = (1, 0, 0, 0)$, we obtain

$$\lambda = \frac{\mathbb{J} \cdot \mathbb{P}}{|\mathbb{P}|}.$$

We have verified in the quantum context that λ is the component of the total angular momentum along the direction of motion.

Now, the effect of a 2π-rotation about the direction of motion is to multiply the state vector by $e^{2\pi i\lambda}$. Thus λ must be a half-integer. A zero-mass state is described by a single component; a doubling of states takes place when, depending on interactions, parity transformations are allowed, changing the sign of the helicity.

3 The free neutral scalar field

It would seem natural to associate elementary particles with unirreps of the Poincaré group. Life is more complicated, however. In practice, Nature seems to love (covariant) differential equations like (2.20), while the representations given by (2.19) are almost never used. Beyond the KG equation, that all free systems satisfy, the solution spaces for other "evolution equations", like the famous Dirac equation, resolve themselves in several invariant subspaces. The representation theory viewpoint should not be too stressed in QFT, which after all tries to grapple with *interacting* systems, and many particles at a time.

The free neutral scalar field is a quantum object satisfying (2.20). We first focus on the equation itself, as a classical one, and then we turn to its meaning in QFT.

3.1 The space of solutions

Let M denote a submanifold of $\mathbb{R}^4$. Twice integrating by parts yields

$$\int_M \phi_1(\Box + m^2)\phi_2\, d^4x = \int_M \phi_2(\Box + m^2)\phi_1\, d^4x + \int_M \operatorname{div} j(\phi_1, \phi_2)\, d^4x,$$

where the vector field j is given by

$$j_\mu(\phi_1, \phi_2) = \phi_1 \partial_\mu \phi_2 - \partial_\mu \phi_1 \phi_2 =: \phi_1 \overleftrightarrow{\partial_\mu} \phi_2.$$

Therefore if both ϕ_1 and ϕ_2 solve the KG equation, we have

$$\partial^\mu j_\mu(\phi_1, \phi_2) = 0.$$

As a consequence, if Σ is a Cauchy hypersurface, and v_1, v_2 are solutions of the KG equation vanishing rapidly enough at "spatial infinity", by the divergence theorem the integral

$$s(v_1, v_2) := \int_\Sigma j_\mu(v_1, v_2)\, d\sigma^\mu = \int_\Sigma v_1 \overleftrightarrow{\partial_\mu} v_2\, d\sigma^\mu$$

does not depend on Σ itself, and defines a symplectic (skewsymmetric, nondegenerate) form on the space V of solutions of the KG equation.

Up to multiples, s is the only Poincaré invariant symplectic form on V. The symplectic space (V, s) we take as our the 1-particle phase space, the starting point for *quantization*.

3.2 The Cauchy problem

Note that the KG equation is of the second order in t, so one must give *two* conditions for solving the Cauchy (initial value) problem for it: the values of the solution and its time derivative at a suitable spacelike surface, say $t = 0$. The so-called Jordan–Pauli function or commutator function $D \equiv D_{JP}$ solves this Cauchy problem. By definition, with $E(\boldsymbol{p}) = \sqrt{m^2 + |\boldsymbol{p}|^2}$,

$$D(x - x') := \frac{1}{(2\pi)^3} \int e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \frac{\sin E(\boldsymbol{p})(t - t')}{E(\boldsymbol{p})}\, d^3p, \tag{3.1}$$

which is then the **integral kernel** on $\mathbb{R}^3$ of $\sin \omega(t - t')/\omega$, a solution of the KG equation characterized by $D(0, \boldsymbol{x}) = 0$ and $\partial_t D(t, \boldsymbol{x})|_{t=0} = \delta(\boldsymbol{x})$ —thus solving the Cauchy problem for it.

Proof. Indeed, denoting $\omega^2 = m^2 - \Delta$, we can rewrite the KG equation as

$$\frac{d}{dt}\begin{pmatrix} v \\ g \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} v \\ g \end{pmatrix} =: A \begin{pmatrix} v \\ g \end{pmatrix}.$$

The (entirely rigorous) solution of this equation with Cauchy data $f(\boldsymbol{x}) \equiv v(0,\boldsymbol{x})$, $h(\boldsymbol{x}) \equiv g(0,\boldsymbol{x})$ is:

$$\begin{pmatrix} v(t,\boldsymbol{x}) \\ g(t,\boldsymbol{x}) \end{pmatrix} = \exp(At) \begin{pmatrix} v(0,\boldsymbol{x}) \\ g(0,\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} [\cos(\omega t) f](\boldsymbol{x}) + [\omega^{-1}\sin(\omega t) h](\boldsymbol{x}) \\ [-\omega\sin(\omega t) f](\boldsymbol{x}) + [\cos(\omega t) h](\boldsymbol{x}) \end{pmatrix}. \tag{3.2}$$

In view of the fact that on momentum space the operator ω is given by multiplication by $E(\boldsymbol{p})$, it is clear that D is the representation of the operator $-\omega^{-1}\sin\omega t$ as an integral kernel on configuration 3-space. $\square$

A more elegant form of the spectral representation (3.1) is

$$D(x) = \frac{i}{(2\pi)^3} \int \operatorname{sgn}(p^0)\delta(p^2 - m^2) e^{-i(px)}\, d^4p. \tag{3.3}$$

To see that this coincides with (3.1), just note that

$$\begin{aligned} \delta(p^2 - m^2) = \delta\big(p_0^2 - E^2(\boldsymbol{p})\big) &= \frac{\delta\big(p_0 - E(\boldsymbol{p})\big)}{2E(\boldsymbol{p})} + \frac{\delta\big(p_0 + E(\boldsymbol{p})\big)}{2E(\boldsymbol{p})} \\ &=: \delta_+(p^2 - m^2) + \delta_-(p^2 - m^2). \end{aligned}$$

As a consequence of $D(0,\boldsymbol{x}) = 0$ and Lorentz invariance we have that the propagator $D(x)$ vanishes for any spacelike argument. Note moreover $\frac{\partial^2 D(x)}{\partial t^2}\big|_{t=0} = 0$. It is instructive to check all this on the explicit expression

$$D(x - x') = \operatorname{sgn}(t - t')\, \frac{\delta\big((x - x')^2\big)}{2\pi}, \tag{3.4}$$

when $m = 0$.

If we are given a differential operator L, acting on functions on spacetime, and if we know the solution $K(x, x')$ of the inhomogeneous problem

$$LK(x, x') = \delta^4(x - x'),$$

then the solution of the inhomogeneous problem with source ρ, namely $L\phi(x) = \rho(x)$, is in principle afforded by

$$\phi(x) = \phi^{\mathrm{h}}(x) + \int K(x, x')\rho(x')\, d^4x',$$

where ϕ^{h} is a solution of the homogeneous equation $L\phi(x) = 0$, perhaps further determined by boundary conditions. Then $K(x, x')$ is called a **propagator** or *proper* Green function for L. Let now K be a solution of the homogeneous equation (such as D):

$$\big(\square + m^2\big)K(x) = 0,$$

fulfilling the conditions

$$\lim_{t\to 0+} K(x,t) = 0 \quad \text{together with} \quad \lim_{t\to 0+} \frac{\partial K(x,t)}{\partial t} = \delta(\boldsymbol{x}).$$

Then we claim that $H(t)K(x)$ is a proper Green function. For the proof:

$$\square\big[H(t)K(x)\big] = \frac{\partial}{\partial t}\Big[H(t)\frac{\partial K(x)}{\partial t} + \delta(t)K(x)\Big] - H(t)\Delta K(x).$$

Because K solves the homogeneous equation, we have to prove

$$2\delta(t)\frac{\partial K(x)}{\partial t} + \delta'(t)K(x) = \delta^4(x).$$

Indeed: from $\delta(t)f(t) = \delta(t)f(0), \delta'(t)f(t) = \delta'(t)f(0) - \delta(t)f'(0)$, plus the initial conditions, the result follows. On the other hand, the difference between two propagators satisfies the homogeneous equation. In our case K is D and $H(t)K(x)$ is called D_{ret}, and then

$$D(x - x') = D_{\text{ret}}(x - x') - D_{\text{adv}}(x - x'),$$

where both D_{ret} and D_{adv} are propagators. The solutions of the homogeneous equation are often called improper Green functions, or even loosely propagators as well.

The first row in the display (3.2), giving the solution of the Cauchy problem for the KG equation with initial conditions $v(0, \boldsymbol{y})$ and $g(0, \boldsymbol{y}) := \frac{\partial v(t,\boldsymbol{y})}{\partial t}\Big|_{t=0}$ as

$$v(t, \boldsymbol{x}) = \int_{\mathbb{R}^3} \Big(D(t, \boldsymbol{x}, 0, \boldsymbol{y})g(0, \boldsymbol{y}) - v(0, \boldsymbol{y})\frac{\partial}{\partial s}\Big|_{s=0} D(t, \boldsymbol{x}; s, \boldsymbol{y})\Big)\, d^3y.$$

Notice that v so given can be directly seen to satisfy $(\Box + m^2)v = 0$ and the initial conditions. The solution is unique because the difference of two solutions would be a solution vanishing together with its derivative on $t = 0$, and therefore vanishing everywhere. By a standard argument, using the fact that D solves the wave equation, the hyperplane $s = 0$ in Minkowski space M_4 can be replaced by any suitable spacelike hypersurface Σ. One obtains:

$$v(x) = \int_\Sigma [D(x, y)\partial^\rho v(y) - v(y)\partial_y^\rho D(x, y)]\, d\sigma_\rho(y).$$

Invoking the symplectic form s, this we rewrite as

$$v(x) = s\big(D(x, .), v(.)\big),$$

or even more abbreviately $v = s(D, v)$. This means that D acts as a "reproducing kernel". In particular:

$$D(x, y) = s\big(D(x, \cdot), D(\cdot, y)\big).$$

3.3 More improper and proper Green functions

We need as well the following distributions (sometimes called Wightman functions):

- The D_+-function:

$$\begin{aligned} D_+(x) &:= \frac{1}{(2\pi)^4}\int_{\mathcal{C}_+} \frac{e^{-i(px)}}{p^2 - m^2}\, d^4p = \frac{i}{(2\pi)^3}\int \delta_+(p^2 - m^2)\, e^{-i(px)}\, d^4p \\ &= \frac{i}{(2\pi)^3}\int \frac{e^{-iE(\boldsymbol{p})t + i\boldsymbol{p}\cdot\boldsymbol{x}}}{2E(\boldsymbol{p})}\, d^3p. \end{aligned} \tag{3.5}$$

  The circuit $\mathcal{C}_+$ turns counterclockwise around the pole at $\Re p_0 > 0$ only. Indeed, by *der Residuensatz*,

$$\frac{1}{2\pi}\int_{\mathcal{C}_+} \frac{e^{-ip^0t}}{p^2 - m^2}\, dp^0 = 2\pi i\Big(\operatorname{Res}\Big|_{p^0=E}\frac{1}{2\pi}\frac{e^{-i(p^0t)}}{p^2 - m^2}\Big) = \frac{i\, e^{-iE(\boldsymbol{p})t}}{2E(\boldsymbol{p})}.$$

- The D_--function

$$D_-(x) := \frac{1}{(2\pi)^4}\int_{\mathcal{C}_-} \frac{e^{-i(px)}}{p^2 - m^2}\, d^4p = \frac{i}{(2\pi)^3}\int \frac{e^{iE(\boldsymbol{p})t + i\boldsymbol{p}\cdot\boldsymbol{x}}}{2E(\boldsymbol{p})}\, d^3p$$
$$= \frac{i}{(2\pi)^3}\int \delta_-(p^2 - m^2)\, e^{-i(px)}\, d^4p.$$

The circuit $\mathcal{C}_-$ now runs clockwise around the pole at $\Re p_0 < 0$ only. *Der Residuensatz* yields now:

$$\frac{1}{2\pi}\int_{\mathcal{C}_-} \frac{e^{ip^0 t}}{p^2 - m^2}\, dp^0 = -2\pi i\Big(\operatorname{Res}\big|_{p^0 = -E} \frac{1}{2\pi}\frac{e^{-i(p^0 t)}}{p^2 - m^2}\Big) = \frac{i\, e^{iE(\boldsymbol{p})t}}{2E(\boldsymbol{p})}.$$

- Thus for the already known commutator function:

$$D(x) = D_+(x) - D_-(x) = \frac{1}{(2\pi)^4}\int_{\mathcal{C}} \frac{e^{i(px)}}{p^2 - m^2}\, d^4p.$$

Here $\mathcal{C}$ turns counterclockwise around both poles in the complex p_0-plane.

Unfortunately, no firm conventions exist on these and similar definitions. We juggle the sundry fourth roots of unit to adapt our notations so that [3, Formulae II.1.10-11] (essentially) hold.

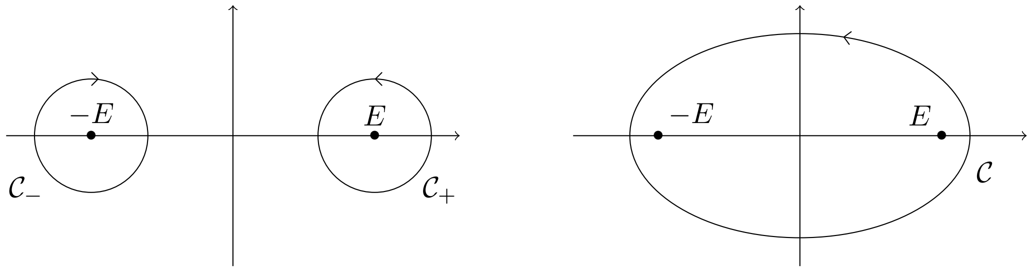

Figure 1: Closed integration contours in the complex p^0-plane

On sees that $D_\pm(x) = D_\mp(-x) = -D^*_\mp(x)$ and $D = \pm 2\,\Re\, D_\pm$. Now $D(x) = -D(-x)$, which by Lorentz invariance implies that D vanishes on spacelike separations, that is, it has support in the (closed) lightcone. We see that

$$D_{\text{ret,adv}}(x) = \pm H(\pm t) D(x) = \pm H(\pm t)\big(D_+(x) - D_-(x)\big).$$

We have for those propagators

$$D_{\text{ret}}(x) = \frac{1}{(2\pi)^4}\int_{\mathcal{C}_r} \frac{e^{-i(px)}}{p^2 - m^2}\, d^4p; \qquad D_{\text{adv}}(x) = \frac{1}{(2\pi)^4}\int_{\mathcal{C}_a} \frac{e^{-i(px)}}{p^2 - m^2}\, d^4p.$$

Here the contour $\mathcal{C}_r, \mathcal{C}_a$ respectively passes the poles in the complex p^0 plane on its left and right. By the same token,

$$\bar{D}(x) := \tfrac{1}{2}\big(D_{\text{ret}}(x) + D_{\text{adv}}(x)\big) = \tfrac{1}{2}\operatorname{sgn}(t) D(x)$$

runs through the poles.

We pause to indicate that in physics, one is interested in the "asymptotic Cauchy problem", in which the Cauchy data are chosen on the surfaces $t = -\infty$ and $t = \infty$. Let us denote the corresponding solutions of the homogeneous equation by $f_{\rm in}, f_{\rm out}$, respectively. Then the solution of the inhomogeneous equation with Lagrangian coupling $\Phi^2 A$ (details forthcoming) is written using the interpolating function

$$f(x) = f_{\rm in}(x) + D_{\rm ret} * A * f(x) = f_{\rm out}(x) + D_{\rm adv} * A * f(x),$$

which implies:

$$f_{\rm out}(x) = \big(1 - D_{\rm adv} * A\big)\big(1 - D_{\rm ret} * A\big)^{-1} * f_{\rm in}(x) =: S_{\rm cl} f_{\rm in}(x),$$

where $S_{\rm cl}$ so defined is the *classical* scattering matrix. It amounts to a symplectic transformation of V.

Also, and foremost, the Stückelberg–Feynman propagator

$$D_F(x) = H(t)D_+(x) + H(-t)D_-(x) = D_{\rm ret}(x) + D_-(x) = D_{\rm adv}(x) + D_+(x).$$

We check

$$D_{\rm ret}(x) + D_-(x) = H(t)D_+(x) - H(t)D_-(x) + D_-(x) = H(t)D_+(x) + H(-t)D_-(x).$$

It holds

$$D_F(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-i(px)}\, d^4p}{p^2 - m^2 - i\varepsilon}.$$

One can define as well a *Dyson propagator* $D_{\bar F}$, corresponding to a contour that runs over the pole with positive real part and below the other pole:

$$\begin{aligned} D_{\bar F}(x) = \frac{1}{(2\pi)^4} \int \frac{e^{-i(px)}\, d^4p}{p^2 - m^2 + i\varepsilon} &= -H(-t)D_+(x) - H(t)D_-(x) \\ = D_{\rm ret}(x) - D_+(x) = D_{\rm adv}(x) - D_-(x) &= D_F^\dagger(x). \end{aligned}$$

The last statement in the sense of integral operators. A prodigious aspect of Feynman and Dyson propagators is the freedom to rotate the contour to the imaginary axis without crossing any of the poles ("Wick rotation"); that allows computing them by integrals on Euclidean space. Also $\bar D = \Re D_F$: historically, this is the link between the Wheeler–Feynman formulation of classical electrodynamics and Feynman's of quantum electrodynamics.

3.3.1 Massless examples

Green functions for massless scalar particles may serve as examples. The reader should be familiar with the Plemelj-Sokhotsky relations $\frac{1}{x \pm i\varepsilon} = \mp i\pi\delta + \mathrm{P}\,\frac{1}{x}$ [4]. For those particles, besides (3.4), we have

$$D_\pm(x - x') = \pm \operatorname{sgn} t\, \frac{\delta((x - x')^2)}{4\pi} - \mathrm{P}\, \frac{i}{4\pi^2 (x - x')^2}.$$

Also,

$$\begin{aligned} D_{\rm ret}(x - x') &= H(t - t') \frac{\delta((x - x')^2)}{2\pi} = H(t - t')\, \frac{\delta\big(|\boldsymbol{x} - \boldsymbol{x}'| - (t - t')\big)}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}, \quad \text{and} \\ D_{\rm adv}(x - x') &= H(t' - t) \frac{\delta((x - x')^2)}{2\pi} = H(t' - t)\, \frac{\delta\big(|\boldsymbol{x} - \boldsymbol{x}'| + (t - t')\big)}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|}. \end{aligned}$$

It follows

$$D_F(x-x') = \frac{\delta((x-x')^2)}{4\pi} - \mathrm{P}\,\frac{i}{4\pi^2(x-x')^2} = \frac{-i}{4\pi^2}\,\frac{1}{(x-x')^2 - i\varepsilon}, \quad \text{and}$$
$$D_{\bar F}(x-x') = \frac{\delta((x-x')^2)}{4\pi} + \mathrm{P}\,\frac{i}{4\pi^2(x-x')^2} = \frac{i}{4\pi^2}\,\frac{1}{(x-x')^2 + i\varepsilon}.$$

The half-advanced, half-retarded propagator

$$\bar D(x-x') = \frac{\delta((x-x')^2)}{4\pi}$$

is the real part of the Feynman and Dyson propagators.

The simplest way to compute these functions *ab initio* is to compute D_+ directly from its definition (3.5). One can as well argue as follows. The formula implies that the Wightman function for *massless* scalars must have the scaling behaviour

$$D_+(\lambda x) = \lambda^{-2} D_+(x).$$

This suggests (with some benefit of hindsight)

$$D_+(x) \propto \frac{1}{(t-i\varepsilon)^2 - |\boldsymbol{x}|^2}\,.$$

Note moreover that

$$D_+(0,\boldsymbol{x}) = \frac{i}{(2\pi)^3}\int \frac{e^{i\boldsymbol{p}\cdot\boldsymbol{x}}}{2|\boldsymbol{p}|}\,d^3p = \frac{i}{(2\pi)^2|\boldsymbol{x}|^2}.$$

Therefore,

$$D_+(x) = \frac{-i}{(2\pi)^2\big((t-i\varepsilon)^2 - |\boldsymbol{x}|^2\big)},$$

as above. For $\zeta = x - i\eta$, with η in the forward lightcone, $D_+(\zeta)$ is analytic.

Exercise 23. Do compute D_+ directly from its definition (3.5).

Exercise 24. Verify the statement on the analicity of $D_+(\zeta)$.

3.4 A new actor

Now consider the symmetric Green function D_1,

$$D_1(x) = \frac{1}{(2\pi)^3}\int \frac{\cos(E(\boldsymbol{p})t - \boldsymbol{p}.\boldsymbol{x})}{E(\boldsymbol{p})}\,d^3p = \frac{1}{(2\pi)^3}\int \delta(p^2-m^2)e^{-i(px)}\,d^4p. \tag{3.6}$$

This is the kernel of the operator $\omega^{-1}\cos\omega t$, a different solution of the KG equation, obeying $D_1(x,y) = D_1(y,x)$. Obviously,

$$D_1(x) = -i\big(D_+(x) + D_-(x)\big); \quad \text{also} \quad D_1 = -2i(D_F - \bar D) = -i(D_F - D_{\bar F}).$$

For the massless case $D_1(x) = -\,\mathrm{P}\,\frac{1}{2\pi^2 x^2}$.

Now, we certainly have:

$$D_1(x,y) = s\big(D(x,.), D_1(.,y)\big),$$

or $D_1 = s(D, D_1)$. Consider as well the operator J given by

$$\exp(A\pi/2) = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}, \quad \text{such that} \quad J^2 = -1_V.$$

We have

$$Jv(x) = \int_\Sigma \Big(D_1(x,y)\partial^\rho v(y)) - v(y)\partial_y^\rho D_1(x,y) \Big)\, d\sigma_\rho(y), \tag{3.7}$$

that is to say $Jv := s(D_1, v)$. This is true because it holds when Σ is the hypersurface $y^0 = 0$. The following distributional identity for the kernels D and D_1 holds as well:

$$D(x,y) = -\int_\Sigma \big(D_1(x,z)\partial_z^\rho D_1(z,y) - \partial_z^\rho D_1(x,z) D_1(z,y) \big)\, d\sigma_\rho(z). \tag{3.8}$$

This is immediate from the operator definition of the kernels, or can be regarded as an exercise in Fourier analysis, using (3.3) and (3.6). We may rewrite this identity as $D = -s(D_1, D_1)$. In effect

$$J^2 v = s(D_1, Jv) = s\big(D_1, s(D_1), v\big) = -v,$$

where (3.8) and interchange of integrations have been used in the last equality.

The operator J is an important object. We pause to spell out how this comes about. Classical 1-particle spaces in the abstract are endowed with no natural complex Hilbert space structure: for neutral scalar particles, only the pair (V, s) of an infinite-dimensional real vector space and a symplectic bilinear form are given. Then one must choose a suitable *complex structure* J for (V, s). Only such a choice gives rise to a complex Hilbert space $\mathcal{H}$, and it constitutes the first step in the quantization process. (In discussions of quantization often the matter of complex structures is omitted, and it is just assumed that $\mathcal{H}$ had been obtained somehow.)

By definition, a *complex structure* J is a real-linear operator on V which satisfies

$$J^2 = -1_V. \tag{3.9}$$

Moreover, we ask for

$$s(Ju, Jv) = s(u,v), \quad \text{for } u, v \in V; \quad \text{and for} \quad s(v, Jv) > 0, \quad \text{for } 0 \neq v \in V.$$

The first condition is that the complex structure be also symplectic; one says that J is *compatible* with the given symplectic form s. The positivity condition is equivalent to demanding that the symmetric bilinear form

$$d_J(u,v) := s(u, Jv)$$

be positive definite on V. Definition (3.9) allows us to regard V as a *complex* vector space under the rule

$$(\alpha + i\beta)v := \alpha v + \beta Jv \qquad \text{for } \alpha, \beta \text{ real},$$

and in this case the *hermitian* form

$$\langle u \mid v \rangle := \langle u \mid v \rangle_J := s(u, Jv) + is(u,v) = d_J(u,v) + is(u,v)$$

is an inner product on V. In conclusion, as hinted at before, a complex structure needs to be given on a space of classical fields before quantization can proceed. The trio (V, s, J) is what constitutes $\big(\mathcal{H}, \langle\cdot \mid \cdot\rangle\big)$. In practice, a slightly less restrictive framework is needed: a real-linear mapping $K : V \to \mathcal{H}$ with dense range such that

$$\langle Ku \mid Kv\rangle := \langle u \mid v\rangle_J. \tag{3.10}$$

So we see that J given by (3.7) is a good complex structure. Compatibility of J with s is also clear:

$$s(Jv_1, Jv_2) = s\big(s(v_1, D_1), s(v_2, D_1)\big) = s\big(s(D, v_1), v_2\big) = s(v_1, v_2).$$

Exercise 25. Prove that D_1 (called the Schwinger "propagator") is given by

$$\frac{1}{(2\pi)^4}\int_{\mathcal{C}_8} \frac{e^{-i(px)}}{p^2 - m^2}\, d^4p;$$

where the closed contour $\mathcal{C}_8$ draws a figure-eight around the poles.

3.5 A more covariant way of life*

For further analysis it is convenient to pass to a fully quadridimensional representation of s (spacetime smearing) and of the action of the propagators. As it turns out, the *same step* is necessary for a good definition of the quantum fields as OVDs. To do so already at the classical level contributes to enhanced understanding of the quantization process.

We need the following definition: a function or distribution on M_4 is of compact support in the past if the intersection of its support with every backward lightcone in M_4 is compact. Analogously is compact support in the future defined. Note that the concept is much weaker than ordinary compactness. It is rather clear that $D^\lambda_{\mathrm{ret}}/D^\lambda_{\mathrm{adv}}$ are convolution inverse powers (in the strict sense) of the KG operator respectively on the class of functions or distributions of compact support in the past/future.

Let h be a smooth function on M_4 of compact support in the past *and* the future. Then

$$v_h(x) = \int D(x, y)h(y)\, d^4y \tag{3.11}$$

is a (smooth) element of V, because $D(., y)$ is a solution of the KG equation. (A good behaviour of h, v_h at spatial infinity is understood.)

Reciprocally, any element $v \in V$ can be represented in this way. For we may take any two spacelike surfaces Σ_1, Σ_2 subject to $\Sigma_1 < \Sigma_2$ and write

$$h_v(y) := \big(\Box + m^2\big)w(y)v(y),$$

where w is a smooth function with $\phi(y) = 0$ before Σ_1 and $\phi(y) = 1$ after Σ_2. Then clearly h_v is of compact support in the past and the future, and we assert $v_{h_v} \equiv v$. In effect, call M the submanifold of M_4 pressed below by Σ_1 and above by Σ_2. By the divergence theorem again

$$\int D(x, y)\big(\Box + m^2\big)[w(y)v(y)]d^4y = \int_{\Sigma_2 - \Sigma_1} \big\{D(x, y)\partial^\nu_y[w(y)v(y)] - w(y)v(y)\partial^\nu D(x, y)\big\}d\sigma_\nu(y).$$

Because w vanish on Σ_1, the second integral vanishes. On Σ_2, on the other hand, $w(y)v(y)$ coincides with $v(y)$, and the conclusion follows.

Of course, such an h_v is far from unique. We next show that can add to the right hand side of the formula defining it any (and only a) function of the form $(\Box + m^2)k$, where k is a smooth function of compact support in the past and the future, but otherwise arbitrary. (In so doing we are identifying elements of V with residue classes of functions of compact support in the past and the future on Minkowski space, modulo the range of the KG operator $\Box + m^2$.) Indeed, if $\int D(x,y)h(y)\,d^4y = 0$, then consider

$$k(x) := \int D_{\mathrm{ret}}(x,y)h(y)\,d^4y = D_{\mathrm{adv}}(x,y)h(y)\,d^4y.$$

This is of compact support in the past, as its support is a closed subset of the future of h; and similarly is of compact support in the future. Now $(\Box + m^2)k = h$. Clearly, h is of compact support in the past and the future as well. For distributional solutions of the KG equation, the same theorems apply, but without the smoothness and decay at spatial infinity restrictions; that is to say, we take any distribution of compact support in the past and the future in the second member of formula (3.11).

Now, simple manipulations, using $D = s(D,D), D_1 = s(D,D_1)$ and interchange of integrations lead to:

$$s(v_1,v_2) = -\int D(x,y)h_{v_1}(x)h_{v_2}(y)\,d^4x\,d^4y; \tag{3.12}$$

as well as

$$Jv(x) = -\int D_1(x,y)h_v(y)\,d^4y \tag{3.13}$$

and

$$d_J(v_1,v_2) := s(v_1,Jv_2) = \int D_1(x,y)h_{v_1}(x)h_{v_2}(y)\,d^4x\,d^4y. \tag{3.14}$$

This expression is positive definite.

Assume now $\operatorname{supp} h_{v_1} \cap$ past of $\operatorname{supp} h_{v_2} = \emptyset$. Recall that $D_F = \bar{D} + \frac{i}{2}D_1$. We obtain

$$\begin{aligned}\langle v_1 \mid v_2\rangle = s(v_1,Jv_2) + is(v_1,v_2) &= \int h_{v_1}(x)[D_1 - iD](x,y)_{v_2}(y)\,d^4x\,d^4y \\ &= -2i\int D_F(x,y)h_{v_1}(x)h_{v_2}(y)\,d^4x\,d^4y.\end{aligned}$$

If $\operatorname{supp} h_{v_1}$ is to the past of $\operatorname{supp} h_{v_2}$,

$$\langle v_2 \mid v_1\rangle = -2i\int D_F(x,y)h_{v_1}(x)h_{v_2}(y)\,d^4x\,d^4y.$$

We perceive that D_F, which plays no classical role, is related to the choice of quantization; hence its inevitability. One can ask: are no other complex structures around equally suitable for quantization? The answer is no, if we ask for Lorentz invariance of J; actually D_1 is uniquely characterized by its employed properties, including positivity and symmetry [5].

Exercise 26. Prove (3.12), (3.13) and (3.14).

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