PLASTICITY. Flow rule for isotropic hardening

For a *perfectly plastic material* we had the following yield hypothesis:

$$f(\sigma_{ij}) = 0, \tag{1}$$

which in the von Mises case became

$$f(\sigma_{ij}) = \sigma_e(\sigma_{ij}) - \sigma_s = \sqrt{\frac{3}{2}s_{ij}s_{ij}} - \sigma_s = 0 \tag{2}$$

For an *isotropically hardening material*, we will, instead, have

$$f\left[\sigma_{ij}, \kappa\left(\epsilon_{ij}^{p}\right)\right] = 0, \tag{3}$$

which in the von Mises case can be written

$$f\left[\sigma_{ij}, \kappa\left(\epsilon_{ij}^{p}\right)\right] = \sigma_e(\sigma_{ij}) - \sigma_f\left(\epsilon_{ij}^{p}\right) = \sqrt{\frac{3}{2}s_{ij}s_{ij}} - \sigma_f\left(\epsilon_{ij}^{p}\right) = 0, \tag{4}$$

where $\sigma_f(\epsilon_{ij}^{p})$ is a scalar (called flow stress) that increases monotonically with the plastic deformation. The von Mises case makes it evident that $\sigma_f(\epsilon_{ij}^{p})$ is the instantaneous plastic yield limit, that we still keep the main diagonal of the stress space as a symmetry axis of the yield surface and that we keep the circular-cylindrical shape of the yield surface. What happens as the plastic flow develops is that since $\sigma_f(\epsilon_{ij}^{p})$ increases, the diameter of the von Mises cylinder increases. See Fig. 1! This is why this plastic hardening is called isotropic hardening.

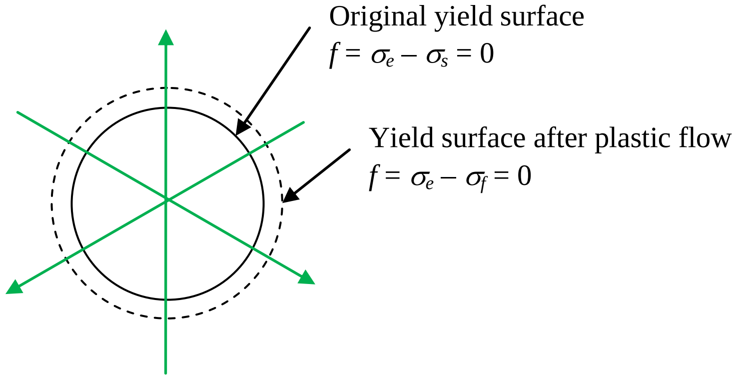

Fig. 1 Isotropic hardening (von Mises)

The most frequent isotropic hardening description results from setting

$$\sigma_f\left(\epsilon_{ij}^{p}\right) = \sigma_f(\epsilon_e^{p}) \tag{5}$$

In case of von Mises,

$$\epsilon_e^p = \int_0^{\epsilon_{ij}^p} d\epsilon_e^p = \int_0^{\epsilon_{ij}^p} \sqrt{\frac{2}{3} d\epsilon_{ij}^p d\epsilon_{ij}^p} \tag{6}$$

(From earlier, we know that the definition of $d\epsilon_e^p$ is such that $dW = \sigma_{ij} d\epsilon_{ij}^p = \sigma_e d\epsilon_e^p$). Note in particular that Eq. (5) shows that one accumulates ϵ_e^p during the whole history of plastic strains (even during unloading, since $d\epsilon_{ij}^p$ is squared in the definition). By this, we can therefore say that $\sigma_f = \sigma_f$[history of ϵ_{ij}^p].

Determination of $d\Lambda$ for the von Mises case

During plastic flow, we must have

$$\begin{cases} f = 0 \\ df = 0 \end{cases} \tag{7}$$

By Eq. (4), Eq. (7b) gives

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} - \frac{\partial f}{d\epsilon_e^p} d\epsilon_e^p = 0 \tag{8}$$

Using the previously derived expression for $\partial f / \partial \sigma_{ij}$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{3}{2} \frac{s_{ij}}{\sigma_e} \tag{9}$$

this gives

$$\frac{3}{2} \frac{s_{ij}}{\sigma_e} d\sigma_{ij} - \frac{\partial \sigma_f}{d\epsilon_e^p} d\epsilon_e^p = 0 \;\Rightarrow\; d\epsilon_e^p = \frac{\frac{3}{2} \frac{s_{ij}}{\sigma_e} d\sigma_{ij}}{\frac{\partial \sigma_f}{d\epsilon_e^p}} \tag{10}$$

The definition of $d\epsilon_e^p$ [for instance, Eq. (6)] and the general flow rule

$$d\epsilon_{ij}^p = d\Lambda \cdot \frac{\partial f}{\partial \sigma_{ij}} \tag{11}$$

gives a simple relation between $d\Lambda$ and $d\epsilon_e^p$:

$$d\epsilon_e^p = \sqrt{\frac{2}{3} d\Lambda \frac{\partial f}{\partial \sigma_{ij}} d\Lambda \frac{\partial f}{\partial \sigma_{ij}}} = d\Lambda \sqrt{\frac{2}{3} \cdot \frac{3}{2} \frac{s_{ij}}{\sigma_e} \cdot \frac{3}{2} \frac{s_{ij}}{\sigma_e}} = d\Lambda \sqrt{\frac{\frac{3}{2} s_{ij} s_{ij}}{\sigma_e^2}} = d\Lambda\,, \tag{12}$$

which gives us the final expression for $d\Lambda$:

$$d\Lambda = \frac{\frac{3}{2} \frac{s_{ij}}{\sigma_e} d\sigma_{ij}}{\frac{d\sigma_f}{d\epsilon_e^p}} \tag{13}$$