PLASTICITY. Flow rule for kinematic hardening

Obviously, for a reversed loading process like the one in the cyclic loading diagram of Fig. 1, the isotropic hardening will lead to a cyclic test behaviour according to the *solid line* OABCDE of Fig. 2 (in which the length of line segment BC is the same as that of line segment AB). It is, however, a well-

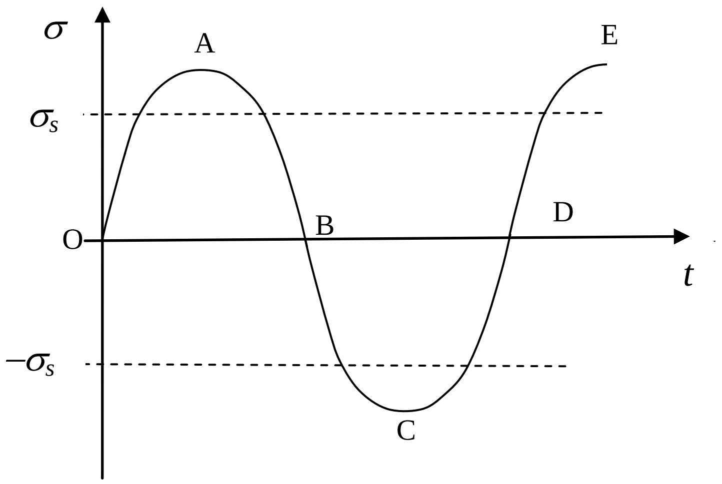

Fig. 1 Cyclic loading

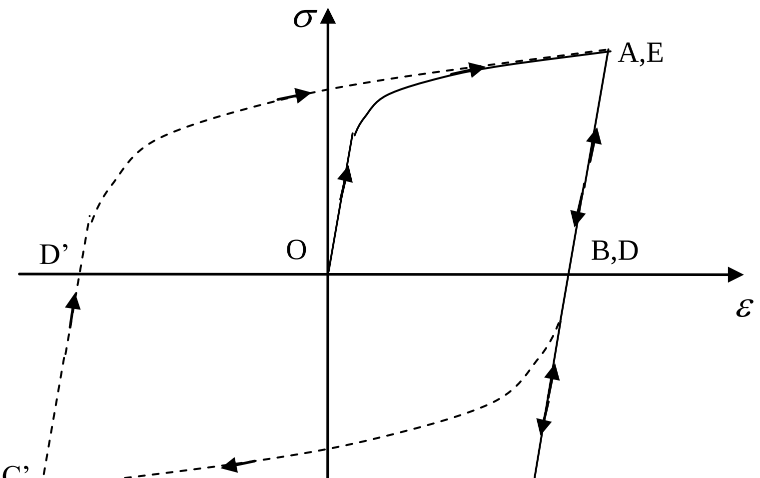

Fig. 2 $\sigma\varepsilon$ diagram of uniaxial cyclic test.
—— Isotropic hardening
- - - - - Kinematic hardening

established fact that in most materials there is a Bauschinger effect, by which a reversed loading will rather follow the *dashed line* OABC'D'E of Fig. 2. This Bauschinger effect can be described by a kinematic hardening in the following way:

$$f = f(\sigma_{ij}, \alpha_{ij}) = \sigma_e(\sigma_{ij} - \alpha_{ij}) - \sigma_s \tag{1}$$

in which α_{ij} is a 2nd order tensor:

$$\alpha_{ij} = \alpha_{ij}(\epsilon_{kl}^p), \tag{2}$$

often called the *backstress*, and σ_s is the yield strength of the virgin material. Since Eq. (1) states that after plastic flow, σ_e will now be computed using $\sigma_{ij} - \alpha_{ij}$ instead of σ_{ij} as argument, we will obviously have a translation of the yield surface. See Fig. 3.

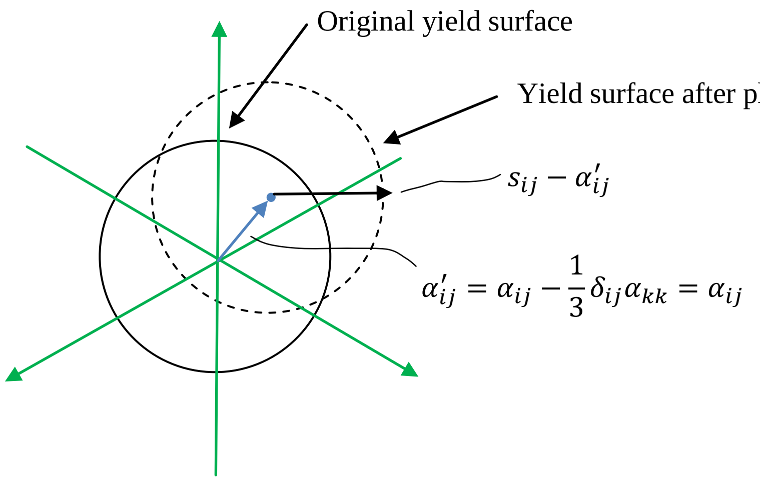

Fig. 3 Example of kinematic hardening (von Mises case)

What remains is, therefore, to establish the function $\alpha_{ij} = \alpha_{ij}(\epsilon_{kl}^p)$. The two most frequent strategies are

$$d\alpha_{ij} = c^{(k)} d\epsilon_{ij}^p \qquad \text{(Prager)} \tag{3}$$

$$d\alpha_{ij} = d\mu(\sigma_{ij} - \alpha_{ij}) \qquad \text{(Ziegler)} \tag{4}$$

where $c^{(k)}$ is a constant that is characteristic for the material (in analogy with $c^{(i)}$ in the isotropic hardening) and $d\mu = d\mu(d\epsilon_{ij}^p)$ is a function of the increment of plastic strain which is also characteristic for the material. To illustrate the difference between the Prager and Ziegler models, we can, for instance, look at the Tresca case shown in Fig. 4. (In the von Mises case, it is, on the other hand, easy to realise that the two models are identical.)

Two important properties of the Prager $d\alpha_{ij}$ may be noticed. Since $c^{(k)}$ is a constant, Eq. (3) can be directly integrated to give

$$\alpha_{ij} = c^{(k)} \epsilon_{ij}^p, \tag{5}$$

and, further,

$$\alpha_{kk} = c^{(k)} \epsilon_{kk}^p \equiv 0 \tag{6}$$

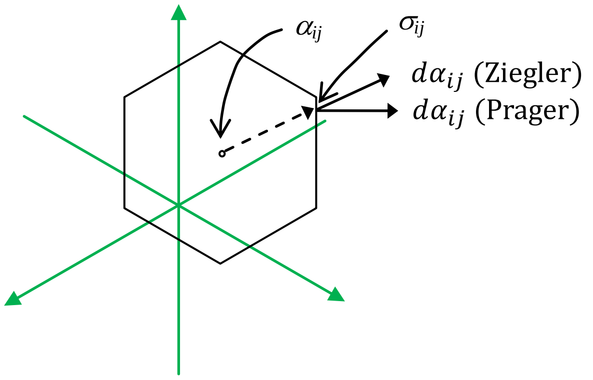

Fig. 4 Prager and Ziegler kinematic hardening shown in a Tresca case

i.e., the Prager α_{ij} is deviatoric:

$$\alpha'_{ij} = \alpha_{ij} \tag{7}$$

General flow rule for kinematic hardening

Again, we start by the consistency condition $df = 0$

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \alpha_{ij}} \frac{\partial \alpha_{ij}}{\partial \epsilon^p_{kl}} d\epsilon^p_{kl} = 0 \tag{8}$$

From the definition of f given in Eq. (1), we can differentiate to find $\partial f / \partial \sigma_{ij}$ and $\partial f / \partial \alpha_{ij}$:

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial \sigma_e}{\partial(\sigma_{ij} - \alpha_{ij})} \frac{\partial(\sigma_{ij} - \alpha_{ij})}{\partial \sigma_{ij}} = \frac{\partial \sigma_e}{\partial(\sigma_{ij} - \alpha_{ij})} \tag{9}$$

$$\frac{\partial f}{\partial \alpha_{ij}} = \frac{\partial \sigma_e}{\partial(\sigma_{ij} - \alpha_{ij})} \frac{\partial(\sigma_{ij} - \alpha_{ij})}{\partial \alpha_{ij}} = -\frac{\partial \sigma_e}{\partial(\sigma_{ij} - \alpha_{ij})} = -\frac{\partial f}{\partial \sigma_{ij}} \tag{10}$$

Eqs. (8) and (10) together with the fundamental normality rule

$$d\epsilon^p_{ij} = d\Lambda \cdot \frac{\partial f}{\partial \sigma_{ij}}, \tag{11}$$

which is still valid, gives

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} - \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial \alpha_{ij}}{\partial \epsilon^p_{kl}} d\epsilon^p_{kl} = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} - \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial \alpha_{ij}}{\partial \epsilon^p_{kl}} d\Lambda \frac{\partial f}{\partial \sigma_{kl}} = 0$$

$$\Rightarrow \quad d\Lambda = \frac{\frac{\partial f}{\partial \sigma_{mn}} d\sigma_{mn}}{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial \alpha_{ij}}{\partial \epsilon^p_{kl}} \frac{\partial f}{\partial \sigma_{kl}}}, \tag{12}$$

and, consequently,

$$d\epsilon_{ij}^{p} = d\Lambda \cdot \frac{\partial f}{\partial \sigma_{ij}} = \frac{\frac{\partial f}{\partial \sigma_{mn}} d\sigma_{mn}}{\frac{\partial f}{\partial \sigma_{pq}} \frac{\partial \alpha_{pq}}{\partial \epsilon_{kl}^{p}} \frac{\partial f}{\partial \sigma_{kl}}} \cdot \frac{\partial f}{\partial \sigma_{ij}} \tag{13}$$

Specialisation to von Mises

With

$$f = \sigma_e(\sigma_{ij} - \alpha_{ij}) - \sigma_s = \sqrt{\frac{3}{2}(s_{ij} - \alpha'_{ij})(s_{ij} - \alpha'_{ij})} \tag{14}$$

we get

$$\begin{aligned} \frac{\partial f}{\partial \sigma_{ij}} &= \frac{\partial f}{\partial s_{kl}} \frac{\partial s_{kl}}{\partial \sigma_{ij}} \\ &= \frac{3}{2\sigma_e}\left[(s_{ij} - \alpha'_{ij}) - \frac{1}{3}\delta_{ij}(s_{mm} - \alpha'_{mm})\right] = \frac{3}{2\sigma_e}(s_{ij} - \alpha'_{ij}) \end{aligned} \tag{15}$$

This inserted into the general kinematic hardening flow rule [Eq. (13)] gives

$$d\epsilon_{ij}^{p} = \frac{(s_{mn} - \alpha_{mn})d\sigma_{mn}}{(s_{pq} - \alpha_{pq}) \frac{\partial \alpha_{pq}}{\partial \epsilon_{kl}^{p}} (s_{kl} - \alpha_{kl})} \cdot (s_{ij} - \alpha_{ij}), \tag{16}$$

where we have also used the property that α_{ij} is deviatoric, *i.e.*, $\alpha'_{ij} = \alpha_{ij}$ (*cf* Eqs. (6) and (7)).

Prager kinematic hardening

Using the Prager hypothesis, Eq. (16) can be simplified. By Eq. (3) we get

$$\frac{\partial \alpha_{pq}}{\partial \epsilon_{kl}^{p}} = \delta_{pk}\delta_{ql}c^{(k)}. \tag{17}$$

This inserted into the flow rule (15) gives

$$\begin{aligned} d\epsilon_{ij}^{p} &= \frac{(s_{mn} - \alpha_{mn})d\sigma_{mn}}{(s_{pq} - \alpha_{pq})\delta_{pk}\delta_{ql}c^{(k)}(s_{kl} - \alpha_{kl})} \cdot (s_{ij} - \alpha_{ij}) \\ &= \frac{(s_{mn} - \alpha_{mn})d\sigma_{mn}}{c^{(k)}(s_{pq} - \alpha_{pq})(s_{pq} - \alpha_{pq})} \cdot (s_{ij} - \alpha_{ij}) \end{aligned} \tag{18}$$