# Derivation of the acoustic wave equation in the presence of gravitational and rotational effects

**Article** *in* The Journal of the Acoustical Society of America · September 1979

DOI: 10.1121/1.383686



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# Derivation of the acoustic wave equation in the presence of gravitational and rotational effects

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(Received 6 October 1978; accepted for publication 30 March 1979)

We derive, from first principles, the multidimensional partial differential equation obeyed by the underwater pressure field in the presence of gravitational and rotational forces acting on the fluid medium. The result is valid for a sound speed which depends on all three spatial dimensions and time. For the special case of a purely depth-dependent sound speed the result reduces essentially to that of Tolstoy. The relationship to the internal wave equation is also presented, as well as other examples, including the effect of Rossby waves.

PACS numbers: 43.30.Bp, 43.20.Bi

## INTRODUCTION

The effects on underwater acoustic propagation of modifications of the sound-speed field have recently been of great interest. The modifications are due to various sources including Rossby waves, several different currents, geostrophic flows, internal tides, and internal waves. Ramsdale[1] has given a concise summary of the extensive research involved as well as doing acoustic field calculations on a perturbation of a parabolic sound-speed profile. In general, however, the various sound-speed modifications, although physically motivated, are *ad hoc* perturbations based on one-dimensional results and analogous extensions thereof. Apparently no systematic derivation of the multidimensional equations satisfied by the acoustic pressure in the presence of gravitational and rotational effects has been given, although in one-dimension, depth, Tolstoy[2,3] has derived equations in the presence of gravitational and rotational effects. In this paper we present the multidimensional derivation and show that Tolstoy's results follow in the special case of one-dimensional sound-speed dependence.

In Sec. I we present the fluid equations and, using standard perturbation methods, the linear equations from which the acoustic equation is derived. Gravitational and rotational effects are included. Section II contains the derivation of the general result, and Sec. III some examples. These latter include an approximation to the wave equation, the Helmholtz equation results for time-independent sound speeds, the Tolstoy results for the special case on one-dimensional sound-speed dependence, and a heuristic derivation of the equation satisfied by the vertical velocity in the absence of acoustic effects, i.e., the internal wave equation. A brief summary is contained in Sec. IV.

## I. FLUID EQUATIONS

In tensor notation the equations describing the interaction of the fluid pressure $p$, the mass density $\rho$, and the three components of fluid velocity $u_j (j=1,2,3)$ are[4,5] firstly the continuity equation

$$\frac{\partial \rho}{\partial t} + \partial_j(\rho u_j) = 0 \, , \tag{1}$$

where $(\partial_1, \partial_2, \partial_3) = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, and the summation convention is assumed. We work in the locally tangent cartesian coordinate frame where the $x$ and $y$ coordinates lie in the plane tangent to the earth's surface and $z$ points into the earth.

Next we have the Euler equations relating the forces acting on the fluid particle to its acceleration. Using Eq. (1) they can be written as

$$\rho \frac{\partial u_j}{\partial t} = -\partial_j p + g\rho\delta_{j3} + f\rho\epsilon_{j3n}u_n \, , \tag{2}$$

where we have included both gravitational and Coriolis forces. Here $g$ is the gravitational acceleration (positive $z$ points into the earth). These gravity forces acting on changes from the mean density in the medium yield motions having a wavelike character. They are called internal waves (IW). Also $\delta_{jm}$ is the Kronecker delta function, $\epsilon_{jmn}$ the antisymmetric third rank tensor, and $f = 2\Omega \sin\theta$ is the inertial frequency, where $\Omega$ is the magnitude of the Earth's angular velocity and $\theta$ the angle of latitude. We can also introduce a component of horizontal rotation by writing $f = f_0 + \beta y$, where $f_0$ and $\beta$ are constants.[6] This is called the $\beta$-plane approximation and is a first approximation of the effect of curvature on the rotating earth. Its introduction leads to the possibility of Rossby waves. In addition, in writing Eq. (2) we have assumed that the vertical motion is small with respect to other velocities, and that the vertical component of the Coriolis force is much less than the gravitational force.

Finally, there is a state equation relating the variables. We write it simply as

$$p = p(\rho) \, . \tag{3}$$

We have not included an explicit dependence of the pressure on temperature or salinity. Instead we assume as is done in practice that this can be included in the sound speed. Equations (1)-(3) are five equations in the five unknowns $p$, $u_j$, and $\rho$.

Using standard perturbation arguments we replace the pressure, density, and velocities as follows:

$$\begin{aligned} p &\to p^0 + \epsilon p \, , \\ \rho &\to \rho^0 + \epsilon\rho \, , \\ u_j &\to \epsilon u_j \, , \end{aligned} \tag{4}$$

where $\epsilon$ is a small parameter and where we've assumed zero hydrodynamic velocity ($u_j^0=0$). The quantities $p$, $\rho$, and $u_j$ will henceforth refer to acoustic variables. Inserting Eq. (4) into Eqs. (1)–(3) and equating coefficients of powers of $\epsilon$ to zero yields, in lowest order, the hydrostatic result $\rho^0=\rho^0(z)$, $p^0=p^0(z)$ and, since we've neglected explicit temperature and salinity gradients, the approximate result

$$c_0^2(z)\frac{d\rho^0}{dz}\approx g\rho^0 , \tag{5}$$

where the lowest order sound speed is defined via

$$\frac{dp^0}{d\rho^0}=c_0^2(z) . \tag{6}$$

The first order (acoustic) results can then be written as

$$\frac{\partial\rho}{\partial t}=-\frac{\partial}{\partial z}(\rho^0u_3)-\rho^0\partial_{j\perp}u_{j\perp} , \tag{7}$$

where the $\partial_{j\perp}$ notation refers to the transverse $(x,y)$ derivative, and $u_{j\perp}$ represents the $(x,y)$ components of the vector $u_j$, viz. $u_j=(u_{j\perp},u_3)$,

$$\frac{\partial p}{\partial z}=-\rho^0\frac{\partial u_3}{\partial t}+g\rho , \tag{8}$$

$$\partial_{j\perp}p=-\rho^0\frac{\partial}{\partial t}u_{j\perp}+f\rho^0\epsilon_{j3m}u_m , \tag{9}$$

and the (linearized) equation of state for the acoustic variables

$$p=c^2(\mathbf{x},t)\rho , \tag{10}$$

where $c(\mathbf{x},t)$ is the full sound speed. Equation (10) follows from an adiabatic compressibility argument. Differentiating Eq. (10) and using Eqs. (7) and (5) yields the result

$$\frac{\partial}{\partial t}\left(\frac{p}{c^2}\right)=-\rho^0\frac{\partial u_3}{\partial z}-\rho^0\partial_{j\perp}u_{j\perp}-g\rho^0c_0^{-2}u_3 . \tag{11}$$

Equations (7), (8), (9), and (11) will be used in the subsequent development. Note that we have included in $c_0(z)$ and $c(\mathbf{x},t)$ the maximal permissible functional dependence consistent with the respective pressure and density variations. The assumptions on $c(\mathbf{x},t)$ will be weakened later in the examples.

## II. DERIVATION OF THE COMBINED EQUATION

Our purpose in this section is to derive a single partial differential equation relating the acoustic pressure alone to the sound speed and the gravitational and rotational parameters. We use Eqs. (7), (8), (9), and (11). First, take $\partial_{j\perp}$ of Eq. (9) and $\epsilon_{3mj}\partial_{m\perp}$ of Eq. (9). This yields the two equations

$$\nabla_\perp^2p=-\rho^0\frac{\partial}{\partial t}\partial_{j\perp}u_{j\perp}+\rho^0f\epsilon_{j3m}\partial_{j\perp}u_{m\perp}+\beta\rho^0u_1 , \tag{12}$$

and

$$f\partial_{j\perp}u_{j\perp}+\beta u_2=\epsilon_{3mj}\frac{\partial}{\partial t}\partial_{m\perp}u_{j\perp} . \tag{13}$$

Next take $\partial/\partial t$ of Eq. (12) and use Eq. (13). The result is

$$\frac{\partial}{\partial t}\nabla_\perp^2p=-\rho^0L(\partial_{j\perp}u_{j\perp})-\beta f\rho^0u_2+\beta\rho^0\frac{\partial}{\partial t}u_1 , \tag{14}$$

where we have defined the operator

$$L=\frac{\partial^2}{\partial t^2}+f^2 . \tag{15}$$

Setting $j=1$ in Eq. (9) and substituting the result in Eq. (14) yields

$$\frac{\partial}{\partial t}\nabla_\perp^2p-\beta\partial_1p=-\rho^0L\partial_{j\perp}u_{j\perp}+2\beta\rho^0\frac{\partial u_1}{\partial t} . \tag{16}$$

The results of $L$ operating on Eqs. (7) and (11), and using Eq. (16), are the equations

$$L\frac{\partial\rho}{\partial t}=-L\frac{\partial}{\partial z}(\rho^0u_3)+\frac{\partial}{\partial t}\nabla_\perp^2p-\beta\partial_1p-2\beta\rho^0\frac{\partial u_1}{\partial t} , \tag{17}$$

and

$$\frac{\partial}{\partial t}\left[\nabla_\perp^2p-L\left(\frac{p}{c^2}\right)\right]-\beta\partial_1p$$
$$=L\left(\frac{\partial}{\partial z}(\rho^0u_3)-g^{-1}N^2(z)\rho^0u_3\right)+2\beta\rho^0\frac{\partial u_1}{\partial t} , \tag{18}$$

where we have defined the Vaisala frequency

$$N^2(z)=g(\rho^0)^{-1}(\rho^0)'-g^2c_0^{-2} , \tag{19}$$

with the prime representing differentiation with respect to $z$. Operating with $L\partial/\partial t$ on Eq. (8) yields

$$\frac{\partial}{\partial t}\left(g\nabla_\perp^2p-L\frac{\partial p}{\partial z}\right)-g\beta\partial_1p$$
$$=L\left(g\frac{\partial}{\partial z}(\rho^0u_3)+\frac{\partial^2}{\partial t^2}(\rho^0u_3)\right)+2\beta g\rho^0\frac{\partial u_1}{\partial t} . \tag{20}$$

Equations (18) and (20) can be rewritten as

$$L\left(g\frac{\partial}{\partial z}(\rho^0u_3)-N^2\rho^0u_3\right)=D , \tag{21}$$

and

$$L\left(g\frac{\partial}{\partial z}(\rho^0u_3)+\frac{\partial^2}{\partial t^2}(\rho^0u_3)\right)=E , \tag{22}$$

where $D$ and $E$ are defined by

$$D=\frac{\partial}{\partial t}\left[g\nabla_\perp^2p-gL\left(\frac{p}{c^2}\right)-2\beta g\rho^0u_1\right]-g\beta\partial_1p , \tag{23}$$

and

$$E=\frac{\partial}{\partial t}\left(g\nabla_\perp^2p-L\frac{\partial p}{\partial z}-2\beta g\rho^0u_1\right)-g\beta\partial_1p . \tag{24}$$

We wish to write $D$ and $E$ as functions of only $p$, so we must solve for $u_1$ in terms of $p$. From Eq. (9) we can derive

$$\frac{\partial^2}{\partial t^2}\partial_1p-f\frac{\partial}{\partial t}\partial_2p=-\rho^0L\frac{\partial u_1}{\partial t} . \tag{25}$$

Operating on Eqs. (21) and (22) with $L$, it is easy to see, using Eq. (25), that $LD$ and $LE$ depend on only $p$. Explicitly they can be written as

$$LD=-gL^2\frac{\partial}{\partial t}\left(\frac{p}{c^2}\right)+\Lambda , \tag{26}$$

and

$$LE=-L^2\frac{\partial}{\partial t}\left(\frac{\partial p}{\partial z}\right)+\Lambda , \tag{27}$$

where

$$\Lambda = gL\frac{\partial}{\partial t}\nabla_{\perp}^2 p - 2\beta g f\frac{\partial}{\partial t}\partial_2 p + g\beta L^*\partial_1 p\ , \tag{28}$$

and $L^* = \partial^2/\partial t^2 - f^2$. Again, multiplying Eqs. (21) and (22) by $L$, and adding and subtracting appropriately multiplied versions of the results yields

$$L^2 R\rho^0 u_3 = LE - LD\ , \tag{29}$$

and

$$gL^2 R\frac{\partial}{\partial z}(\rho^0 u_3) = \frac{\partial^2}{\partial t^2}LD + N^2 LE\ , \tag{30}$$

where the operator $R$ is defined by

$$R = \frac{\partial^2}{\partial t^2} + N^2\ . \tag{31}$$

Multiplying Eq. (29) by the operator $g\partial/\partial z$ and combining the result with Eq. (30) yields, after some algebra, a single equation involving only the pressure and given by

$$R^2\Lambda = L^2\frac{\partial}{\partial t}\left\{R\left[g\frac{\partial^2}{\partial t^2}\left(\frac{p}{c^2}\right) + N^2\frac{\partial p}{\partial z}\right] + g\left(R\frac{\partial}{\partial z} - (N^2)'\right)\left(\frac{gp}{c^2} - \frac{\partial p}{\partial z}\right)\right\}. \tag{32}$$

Equation (32) is the final result for the general sound speed $c(\mathbf{x},t)$ without neglecting any terms in the derivation. To our knowledge it has not appeared before. It is much too complicated to be useful, but is simplified considerably in the examples below.

## III. EXAMPLES

### A. Example 1

For $\beta = 0$, Eq. (32) can be written as

$$L\frac{\partial}{\partial t}T = 0\ , \tag{33}$$

where $T$ is defined as

$$T = gR^2\nabla_{\perp}^2 p - LR\left[g\frac{\partial^2}{\partial t^2}\left(\frac{p}{c^2}\right) + N^2\frac{\partial p}{\partial z}\right] - gL\left(R\frac{\partial}{\partial z} - (N^2)'\right)\left(\frac{gp}{c^2} - \frac{\partial p}{\partial z}\right). \tag{34}$$

In addition if we neglect all terms involving powers of $g$ higher than one, we can approximately write

$$T \cong gRU\ , \tag{35}$$

where

$$U = R\nabla_{\perp}^2 p + L\rho^0\frac{\partial}{\partial z}\left[(\rho^0)^{-1}\frac{\partial p}{\partial z}\right] - L\frac{\partial^2}{\partial t^2}\left(\frac{p}{c^2}\right). \tag{36}$$

A sufficient condition for a solution of Eqs. (33) and (35) is $U = 0$, which yields an equation closely related to the usual acoustic wave equation. The only difference can be thought of as a horizontal scaling.

### B. Example 2

Assume the sound speed is time independent, i.e.,

$$c(\mathbf{x},t) = c_2(\mathbf{x})\ . \tag{37}$$

Then we can introduce the Fourier transform in time with the notation ($\omega$ is acoustic frequency)

$$\tilde{p}(\mathbf{x},\omega) = \int \exp(i\omega t)p(\mathbf{x},t)\,dt\ , \tag{38}$$

and, using the approximation $c_2^{-2} - c_0^{-2} \approx 0$, Eq. (32) can be transformed to yield the equation

$$\begin{aligned}\rho^0\frac{\partial}{\partial z}\left((\rho^0)^{-1}\frac{\partial\tilde{p}}{\partial z}\right) &+ (\omega^2 - N^2)^{-1}(N^2)'\frac{\partial\tilde{p}}{\partial z} \\ &+ (\omega^2 - N^2)(\omega^2 - f^2)^{-1}\nabla_{\perp}^2\tilde{p} + 2\beta f(\omega^2 - N^2)(\omega^2 - f^2)^{-2}\frac{\partial\tilde{p}}{\partial y} \\ &+ i\beta\omega^{-1}(\omega^2 - N^2)(\omega^2 + f^2)(\omega^2 - f^2)^{-2}\frac{\partial\tilde{p}}{\partial x} \\ &+ \left(\omega^2 c_2^{-2} + 2gc_2^{-3}\frac{\partial c_2}{\partial z} - gc_2^{-2}(\omega^2 - N^2)^{-1}(N^2)'\right)\tilde{p} = 0\ .\end{aligned} \tag{39}$$

Eliminating the gravitational term ($g = 0$) and assuming that the acoustic frequency is much larger than the inertial frequency ($\omega \gg f$), Eq. (39) becomes

$$\rho^0\frac{\partial}{\partial z}\left((\rho^0)^{-1}\frac{\partial\tilde{p}}{\partial z}\right) + \nabla_{\perp}^2\tilde{p} + \omega^2 c_2^{-2}\tilde{p} + 2\beta f\omega^{-2}\frac{\partial\tilde{p}}{\partial y} + i\beta\omega^{-1}\frac{\partial\tilde{p}}{\partial x} = 0\ , \tag{40}$$

which reduces to the familiar three-dimensional Helmholtz equation if we also set the component of horizontal rotation to zero ($\beta = 0$).

### C. Example 3

Assume the sound speed depends on only depth, i.e.,

$$c(\mathbf{x},t) = c_0(z)\ , \tag{41}$$

and also set $\beta = 0$ so we can introduce an additional transverse spatial Fourier transform, viz. ($K_\perp$ is the transverse wavenumber)

$$\bar{p}(z,K_\perp,\omega) = \int \exp(-iK_\perp\cdot x_\perp)\tilde{p}(\mathbf{x},\omega)\,dx_\perp\ . \tag{42}$$

Then Eq. (39) reduces to the one-dimensional equation

$$\begin{aligned}\rho^0[(\rho^0)^{-1}\bar{p}']' &+ (\omega^2 - N^2)^{-1}(N^2)'\bar{p}' \\ &+ [\omega^2 c_0^{-2} + K_\perp^2(N^2 - \omega^2)(\omega^2 - f_0^2)^{-1} \\ &\quad + 2gc_0^{-3}c_0' - gc_0^{-2}(N^2)'(\omega^2 - N^2)^{-1}]\bar{p} = 0\ ,\end{aligned} \tag{43}$$

which is similar to an equation derived by Tolstoy,[2,3] but where we have retained additional terms.[7] Again, for $g = 0$ and $\omega \gg f_0$, Eq. (43) becomes the familiar one-dimensional Helmholtz equation

$$\rho^0[(\rho^0)^{-1}\bar{p}']' + (\omega^2 c_0^{-2} - K_\perp^2)\bar{p} = 0\ . \tag{44}$$

In Eq. (43) it is easiest to see the modification of the sound speed introduced by the presence of gravitational and rotational effects. We refer to Tolstoy[2,3] for a thorough discussion of the analytic properties of equations of this type.

### D. Example 4: Internal wave equation

A brief heuristic derivation of the spatial internal wave equation can also be given using our previous results. In Eq. (11) use the result for an incompressible fluid, $\partial_j u_j = 0$, replace $c$ by $c_2$, and use the approximation

$c_2 c_0^{-1} \approx 1$. Equation (11) then becomes

$$\frac{\partial p}{\partial t} = -g\rho^0 u_3 . \tag{45}$$

Fourier transform Eq. (45) in time ($w$ is the internal wave frequency; the tilde notation is defined in Example 2) to yield

$$\tilde{p} = -igw^{-1}\rho^0 \tilde{u}_3 . \tag{46}$$

Substitute Eq. (46) into Eq. (39) and let $c_2 \to \infty$ (incompressible fluid). The result is

$$\begin{aligned}(\rho^0)^{-1}\frac{\partial}{\partial z}\left(\rho^0 \frac{\partial \tilde{u}_3}{\partial z}\right) &+ (N^2)'(w^2 - N^2)^{-1}\frac{\partial \tilde{u}_3}{\partial z} \\ &+ (w^2 - N^2)(w^2 - f^2)^{-1}\nabla_\perp^2 \tilde{u}_3 \\ &+ \{[(\rho^0)^{-1}(\rho^0)']' + (\rho^0)^{-1}(\rho^0)'(N^2)'(w^2 - N^2)^{-1}\}\tilde{u}_3 \\ &+ \beta(w^2 - N^2)(w^2 - f^2)^{-2}\left(2f\frac{\partial \tilde{u}_3}{\partial y} + iw^{-1}(w^2 + f^2)\frac{\partial \tilde{u}_3}{\partial z}\right) = 0 .\end{aligned} \tag{47}$$

An additional transverse spatial Fourier transform of Eq. (47) yields, for $\beta = 0$ ($k_\perp$ is the transverse wavenumber),

$$\begin{aligned}(\rho^0)^{-1}(\rho^0 \bar{u}_3')' &+ g(w^2 - N^2)^{-1}[(\rho^0)^{-1}(\rho^0)']'\bar{u}_3' \\ &+ \{k_\perp^2(N^2 - w^2)(w^2 - f_0^2)^{-1} + [(\rho^0)^{-1}(\rho^0)']' \\ &\quad \times [1 + g(\rho^0)^{-1}(\rho^0)'(w^2 - N^2)^{-1}]\}\bar{u}_3 = 0 .\end{aligned} \tag{48}$$

Replacing the explicit density terms in Eq. (48) by a constant yields a result equivalent to the Boussinesq approximation,[8] under which assumption the IW equation is usually derived. The result is

$$\bar{u}_3'' + k_\perp^2(N^2 - w^2)(w^2 - f_0^2)^{-1}\bar{u}_3 = 0 , \tag{49}$$

which is the usual IW equation in one dimension. An extensive discussion of solvable examples of this equation can be found in the book by Roberts.[9] Finally, we remark that although our discussion of this example has been brief, the results can of course be derived more formally starting with the fluid equations and neglecting the density variation in the inertial terms.[10]

## IV. SUMMARY

We have presented the derivation of the partial differential equation satisfied by the acoustic pressure in the presence of gravitational and rotational effects. The multidimensional results we present apparently have not appeared before. For the special case of one-dimensional variation, the results reduce essentially to those due to Tolstoy. A brief outline of some of these results was presented elsewhere.[11]


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[3]I. Tolstoy, *Wave Propagation* (McGraw–Hill, New York, 1973), Chap. 4.

[4]W. Krauss, *Methods and Results of Theoretical Oceanography. I. Dynamics of the Homogeneous and Quasihomogeneous Ocean* (Gebrueder Borntraeger, Berlin, 1973), Chap. 1.

[5]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison–Wesley, Reading, MA, 1959), Chap. 1.

[6]See Ref. 3, p. 161 and Ref. 4, p. 38.

[7]Tolstoy's equations (Ref. 2, pp. 211–212 or Ref. 3, p. 143) are written in terms of displacements, ours in terms of pressure. In addition we have kept terms depending on $c_0'$ and $N'$ which Tolstoy neglects. The net results are additional modifications of the dispersion relation.

[8]O. M. Phillips, *The Dynamics of the Upper Ocean* (Cambridge U. P., Cambridge, 1969), p. 14.

[9]J. Roberts, *Internal Gravity Waves in the Ocean* (Marcel Dekker, New York, 1975), Sec. 3.4.

[10]See Ref. 3, p. 44.

[11]J. A. DeSanto, "Theoretical Methods in Ocean Acoustics" in *Ocean Acoustics*, edited by J. A. DeSanto, Vol. 8 of Topics in Current Physics (Springer, Heidelberg, 1979), pp. 15–18.